Research Article

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New family of time series models and its bayesian analysis

Synoptic Abstract

A new family of time series models, called the Full Range Autoregressive model, is introduced which avoids the difficult problem of order determination in time series analysis. Some of the basic statistical properties of the new model are studied. Further, the paper describes the Bayesian inference and forecasting as applied to the Full Range Autoregressive model. The Canadian lynx data is used to compare the efficiency of the predictive power of the new model with those of some of the existing models in the time series literature.

Keywords: full range autoregressive model, identifiability, stationary condition, posterior distribution, bayesian predictive distribution

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Introduction

The early days of time series analysis, most of the models fitted to the real life data were restricted to low orders because of availability of high speed computers and other facilities. However, now with the availability of high speed computers, there is no need for this type of restriction on the order determination and estimation of the fitted models. Further, most of the work in time series analysis are concerned with series having the property that the degree of dependence between observations, separated by a long time span, is zero or highly negligible. However, the empirical studies by Lawrance and Kottegoda[1] reveal, particularly in cases arising in economics and hydrology, that the degree of dependence between observations a long time span apart, though small, is by no means negligible. Therefore, there is still a need for a family of models which can fully depict the properties of stationarity, linearity and long range dependence.

Moreover, the existing theory of autoregressive models assume that the coefficients of the model are not connected in any way among each other. Therefore, it would be useful, from practical point of view, to propose new models, called the Full Range Auto Regressive model and denoted as FRAR model for short, which can accommodate long range dependence and have the property that the coefficients of the past values in the model are functions of a limited number of parameters.

Thus, the chief objective of this paper is to introduce a family of new models which would involve only a few parameters and at the same time incorporate long range dependence, which would be an acceptable alternative to the current models representing stationary time series.

A family of models, introduced in this paper, called Full Range Auto Regressive model and denoted as FRAR model for short, are defined in such a way that they possess the following basic features.

i. The models should be capable of representing long term persistence. This is justified by the fact that the future may not depend on the present and a few past values alone, but may depend on the present and the whole past.

ii. The parameters of the model, which are likely to be large in number due to (1), should exhibit some degree of dependence among themselves.

Therefore, the new models are expected to have infinite structure with a finite number of parameters and so completely avoid the problem of order determination.

An outline of this paper is as follows. In Section 2, the FRAR model is defined, the identifiability region is obtained, the stationarity condition is derived, and the asymptotic stationarity is studied. In Section 3, the Bayesian analysis of the FRAR model is discussed and the predictive density of a single future observation is derived. In Section 4 the Canadian lynx data is used for forecasting through the FRAR model. In Section 5 a comparative study is provided to examine the efficiency of FRAR model. In Section 6 the summary and conclusion is given.

The full range autoregressive model

The model

We define a family of models by a discrete-time stochastic process (X_t), $t=0,\pm1,\pm2,...$, called the Full Range Auto Regressive (FRAR) model, by the difference equation

$$X_t = \sum_{r=1}^{\infty} a_r X_{t-r} + e_t \qquad (1)$$

where $a_r = k\sin(r\theta)\cos(r\phi)/\alpha^r$, $(r=1,\ 2,\ 3,...)$, k, α, θ and ϕ are parameters, e_1, e_2, e_3, … are independent and identically distributed normal random variables with mean zero and variance σ^2. The initial assumptions about the parameters are as follows:

It is assumed that X_t will influence X_{t+n} for all positive n and the influence of X_t on X_{t+n} will decrease, at least for large n, and become insignificant as n becomes very large, because more important for the recent observations and less important for an older observations. Hence a_n must tend to zero as n goes to infinity. This is achieved by assuming that $\alpha>1$. The feasibility of X_t having various magnitudes of influence on X_{t+n}, when n is small, is made possible by allowing k to take any real value. Because of the periodicity of the circular functions sine and cosine, the domain of θ and ϕ are restricted to the interval $[0,\ 2\pi)$.

Thus, the initial assumptions are $\alpha>1$, $k\in R$, and θ, $\phi\in[0,\ 2\pi)$. i.e., $\Theta=(\alpha,k,\theta,\phi)\in S^*$, where

$S^* = \{\alpha, k, \theta, \phi \mid k \in R, \alpha > 1, \theta, \phi \in [0, 2\pi)\}$. Further restrictions on the range of the parameters are placed by examining the identifiability of the model.

Identifiability condition

Identifiability ensures that there is a one to one correspondence between the parameter space and set of associated probability models. Without identifiability it is meaningless to proceed to estimate the parameters of a model using a set of given data. In the present context, identifiability is achieved by restricting the parameters space in such a way that no two points in the parameter space could produce the same time series model.

The coefficients a_n's in (1) are functions of k, α, θ, ϕ as well as n. That is, $a_n = a_n(k, \alpha, \theta, \phi) = k\sin(n\theta)\cos(n\phi)/\alpha^n$, $\theta \in S^*$, $n=1, 2, 3, \ldots$.

Define $A = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, \pi \le \theta, \phi < 2\pi\}$,

$$B = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, 0\le\theta<\pi, \pi\le\phi<2\pi\}, \quad (2)$$

$$C = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, \pi\le\theta<2\pi, 0\le\phi<\pi\},$$

$$D = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, 0\le\theta, \phi<\pi\}.$$

Since $a_n = a_n(k, \alpha, \theta, \phi) = a_n(-k, \alpha, 2\pi-\theta, 2\pi-\phi)$, $\theta \in S^*$ to each $(\alpha, k, \theta, \phi)$ belonging to A there is a $(\alpha, k, \theta', \phi')(\theta'=2\pi-\theta \text{ and } \phi'=2\pi-\phi)$ belonging to D such that $a_n(k, \alpha, \theta, \phi) = a_n(-k, \alpha, \theta', \phi')$. So A is omitted. Similarly, it can be shown that B and C can also be omitted.

Define $D_1 = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, \pi/2\le\theta, \phi<\pi\}$,

$$D_2 = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, 0\le\theta<\pi/2, \pi/2\le\phi<\pi\},$$

$$D_3 = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, 0\le\theta, \phi<\pi/2\},$$

$$D_4 = \{\alpha, k, \theta, \phi \mid \alpha>1, k\in R, \pi/2\le\theta<\pi, 0\le\phi<\pi/2\}.$$

Since $a_n(k, \alpha, \theta, \phi) = a_n(-k, \alpha, \pi-\theta, \pi-\phi)$ for

$$k \in R, \ \alpha > 1, \ 0 \le \theta, \ \phi < \pi \quad (3)$$

Using (3) it can be shown as before, that the regions D_1 and D_2 can be omitted. Since no further reduction is possible, it is finally deduced that the region of identifiability of the model is given by $S = \{\alpha, k, \theta, \phi \mid k\in R, \alpha>1, \theta\in[0,\pi), \phi\in[0,\pi/2)\}$.

Stationarity of the FRAR process

The stationarity of the newly developed FRAR time series model is now examined. The model is given by $X_t = \sum_{r=1}^{\infty} a_r X_{t-r} + e_t$ That is, $(1 - a_1 B - a_2 B^2 - \ldots) X_t = e_t$, where B is the backward shift operator, defined by $B^n X_t = X_{t-n}$. Thus, the model is given by $\Psi(B) X_t = e_t$, or $X_t = \Psi^{-1}(B) e_t$, where $\Psi(B) = 1 - a_1 B - a_2 B^2 - \ldots$.

Box and Jenkins[2] and Priestley[3] have shown that a necessary condition for the stationarity of such processes is that the roots of the equation $\Psi(B)=0$ must all lie outside the unit circle. So, it is now proposed to investigate the nature of the zeros of $\Psi(B)$.

The power series $\Psi(B)$ may be rewritten as $\Psi(B) = 1 - [a_1 B + a_2 B^2 + \ldots] = 1 - \sum_{n=1}^{\infty} a_n B^n$, where

$$a_n B^n = (kB^n/\alpha^n)[\sin(n\theta)\cos(n\phi)] = (k' B^n/\alpha^n)[\sin(n\theta_1) + \sin(n\theta_2)]$$

$k' = k/2$, $\theta_1 = \theta + \phi$ and $\theta_2 = \theta - \phi$.

Therefore, $\sum_{n=1}^{\infty} a_n B^n = \sum_{n=1}^{\infty} \frac{k' B^n}{\alpha^n} \sin(n\theta_1) + \sum_{n=1}^{\infty} \frac{k' B^n}{\alpha^n} \sin(n\theta_2)$.

The above two series are separately evaluated below.

$$\sum_{n=1}^{\infty} \frac{k' B^n}{\alpha^n} \sin(n\theta_1) = IP \text{ of } \sum_{n=1}^{\infty} \frac{k' B^n}{\alpha^n} e^{in\theta_1} = IP\left\{kBe^{i\theta_1}\left(\alpha - Be^{i\theta_1}\right)^{-1}\right\} = k' B\alpha \sin(\theta_1)/G_1$$

where $G_1 = B^2 + \alpha^2 - 2B\alpha\cos(\theta_1)$ and *IP* stands for imaginary part.

Similarly, it can be shown that $\sum_{n=1}^{\infty} \frac{k' B^n}{\alpha^n} \sin(n\theta_2) = (k' B\alpha \sin(n\theta_2))/G_2$, where $G_2 = B^2 + \alpha^2 - 2B\alpha\cos(\theta_2)$.

Therefore,

$$\sum_{n=1}^{\infty} a_n B^n = k' B\alpha\left[(B^2+\alpha^2)(\sin(\theta_1)+\sin(\theta_2)) - 2B\alpha(\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2))\right]/G_1 G_2$$

Thus, $\Psi(B) = 1 - \sum_{n=1}^{\infty} a_n B^n = 0$ implies that

$$(B^2+\alpha^2-2B\alpha\cos(\theta_1))(B^2+\alpha^2-2B\alpha\cos(\theta_2)) - k' B\alpha\left[(B^2+\alpha^2)s_1 - 2Bd_1c_2\right] = 0$$

where $c_1 = \cos(\theta_1) + \cos(\theta_2) = 2\cos(\theta)\cos(\varphi)$, $c_2 = \sin(2\theta_2)$, $s_1 = \sin(\theta_1) + \sin(\theta_2) = 2\sin(\theta)\cos(\varphi)$, $d_1 = \cos(\theta_1) - \cos(\theta_2)$.

After simplifying, the above equation becomes $B^4 - B^3\alpha(2c_1 + k's_1) + B^2\alpha^2(2 + 4d_1 + 2k'c_2) - B\alpha^3(2c_1 + k's_1) + \alpha^4 = 0$. Thus,

$$B^4 - B^3\alpha A_1 + B^2\alpha^2 A_2 - B\alpha^3 A_1 + \alpha^4 = 0 \quad (4)$$

or

$$S^4 - A_1 S^3 + A_2 S^2 - A_1 S + 1 = 0 \quad (5)$$

where ,

$A_2 = 2 + 4d_1 + 2k'c_2 = 2[1 - \sin(\phi)(4\sin(\theta) - k\cos(\phi))]$, and $S = B/\alpha$.

This equation (of degree 4) reduces to $Z^2 - A_1 Z + (A_2 - 2) = 0$ where $Z = S + (1/S)$.

The roots of this equation are, say r_1 and r_2, are given by $Z = (1/2)\left[A_1 \pm \sqrt{(A_1^2 - 4A_2 + 8)}\right]$.

Since $Z = S + (1/S)$, one finally gets the four roots of the equation (4), as $R_1 = (1/2)\left[r_1 + \sqrt{r_1^2 - 4}\right]$, $R_2 = (1/2)\left[r_1 - \sqrt{r_1^2 - 4}\right]$, $R_3 = (1/2)\left[r_2 + \sqrt{r_2^2 - 4}\right]$ and $R_2 = (1/2)\left[r_2 - \sqrt{r_2^2 - 4}\right]$.

The equation (5) implies that, if S_0 is a root of the equation (5) then $1/S_0$ is also a root. This implies that αS_0 and (α/S_0) are roots of equation (4). Therefore the process is stationary for sufficiently large values of α. But when α is small it seems difficult to examine the stationarity of the process by this approach. Hence, it is proposed to study the asymptotic stationarity of the process in the following section.

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Asymptotic stationarity of the FRAR process

In this section we derive the condition for asymptotic stationarity of the FRAR process. For which one has to solve the difference equation (1), so as to obtain an expression for X_t in terms of e_t, e_{t-1}, e_{t-2}, e_{t-3}, ... The precise solution of this equation depends on the initial conditions. So to investigate the nature of the first and second moments of X_t, following Priestley [3], it is assumed that $X_t=0$ for $t<-N$, N being the number of observations in the time series. Then solving (1) by repeated substitutions one obtains

$$X_t=e_t+a_{11}X_{t-1}+a_{12}X_{t-2}+a_{13}X_{t-3}+\dots,$$

where $a_{1j}=a_j=\left(k/\alpha^j\right)\sin(j\theta)\cos(j\phi)$; $j=1, 2, \dots$,

$$=e_t+a_{11}e_{t-1}+a_{22}X_{t-2}+a_{23}X_{t-3}+a_{24}X_{t-4}\dots,$$

where $a_{2j}=a_{11}a_{1j-1}+a_{1j}$; $j=2, 3, 4 \dots$.

Similarly proceeding one finally gets

$$X_t=\left[e_t+a_{11}e_{t-1}+a_{22}e_{t-2}+a_{33}e_{t-3}+a_{44}e_{t-4}+\dots+a_{pp}e_{t-p}\right]+\left[a_{p+1p+1}X_{t-(p+1)}+a_{p+1p+2}X_{t-(p+2)}+\dots\right]$$

where $a_{ij}=a_{i-1i-1}a_{1j+1-i}+a_{i-1j}$ with $j>i=2, 3, 4 \dots$. Thus, if it is assumed that $X_t=0$ for $t\le -N$, which implies has $n=N+t-1$, then, $X_t=e_t+a_{11}e_{t-1}+a_{22}e_{t-2}+a_{33}e_{t-3}+a_{44}e_{t-4}+\dots+a_{N+t-1,N+t-1}e_{1-N}$.

Further, it can be shown that

$$E[X_tX_{t+1}]=\sigma_e^2\left[a_{11}\left(1+a_{11}^2+a_{22}^2+\dots+a_{N+t-1\ N+t-1}^2\right)+\left(a_{11}a_{12}+a_{22}a_{23}+\dots+a_{N+t-2\ N+t-2}a_{N+t-2\ N+t-1}\right)\right]$$

$$E[X_tX_{t+2}]=\sigma_e^2\left[a_{22}\left(1+a_{11}^2+a_{22}^2+\dots+a_{N+t-1\ N+t-1}^2\right)+a_{11}\left(a_{11}a_{12}+a_{22}a_{23}+\dots+a_{N+t-2\ N+t-2}a_{N+t-2\ N+t-1}\right)+\left(a_{11}a_{13}+a_{22}a_{24}+\dots+a_{N+t-3\ N+t-3}a_{N+t-3\ N+t-1}\right)\right]$$

$$E[X_tX_{t+3}]=\sigma_e^2\left[a_{33}\left(1+a_{11}^2\left(1+a_{22}+a_{33}+\dots+a_{N+t-3\ N+t-3}^2\right)\right)+a_{11}\left(a_{22}a_{34}+a_{33}a_{45}+\dots+a_{N+t-3\ N+t-3}a_{N+t\ N+t-1}\right)+\left(a_{11}a_{34}+a_{22}a_{45}+\dots+a_{N+t-1\ N+t-1}a_{N+t-3\ N+t-3}\right)\right]$$ and in general

$$E[X_tX_{t+s}]=\sigma_e^2\left[a_{ss}+a_{11}a_{s+1s+1}+a_{22}a_{s+2s+2}+\dots+a_{N+t-1\ N+t-1}a_{N+t+s-1\ N+t+s-1}\right]$$

where $a_{ss}=a_{11}a_{s-1s-1}+a_{s-1s}$. Therefore, allowing, we get $E\left[X_t\right]=0$, $Var[X_t]=\sigma_e^2\left[1+a_{11}^2+a_{22}^2+\dots\right]$ and $E[X_tX_{t+s}]=\sigma_e^2\left[a_{ss}+a_{11}a_{s+1s+1}+\dots\right]$ provided the series on the right converges. Thus, it is seen that if $E[X_tX_{t+s}]$ exists then it is a function of s only. In order to examine the convergence of $Var[X_t]$ and $E[X_tX_{t+s}]$, first the behaviour of a_{ij}, as j tends infinity, is investigated. Since $a_{1j}=a_j=\left(k/\alpha^j\right)\sin(j\theta)\cos(j\varphi)$, $|a_{1j}|\le|k|/\alpha^j$. Similarly, $|a_{2j}|\le|k|(1+|k|)/\alpha^j$; $j\ge 2$. Thus, in general $|a_{nj}|\le|k|(1+|k|)^{n-1}/\alpha^j$, for $j\ge n$.

Since $\alpha>1$, the above relation implies that $|a_{nj}|\to 0$ as $j\to\infty$, for any fixed n. Thus $\sum_{n=1}^{\infty}a_{jj}^2$ will converge if $\left|\frac{(1+|k|)}{\alpha}\right|<1$.

If we assume that $1-\alpha<k<\alpha-1$, then one can show that $Var[X_t]=\sigma_{X_t}^2\le\sigma_e^2\frac{k^2}{(1+k)^2}\left[\frac{\alpha^2}{\alpha^2-(1+k)^2}\right]$ and $E[X_tX_{t+s}]\le\sigma_e^2\frac{k^2}{(1+k)^2}\frac{k(1+k)^{s-1}}{\alpha^s}\left[\frac{\alpha^2}{\alpha^2-(1+k)^2}\right]$.

Therefore, the auto-correlation function of the process exists and, as shown earlier, it is a function of s only. Finally allowing $t\to\infty$, it is seen that

i) $\lim_{t\to\infty}E[X_t]$ and $\lim_{t\to\infty}Var[X_t]$ exist finitely;

ii) $\lim_{t\to\infty}Cov[X_t,X_{t+s}]$ exists finitely and is a function of 's' only.

Thus, the condition for $\{X_t\}$ to be asymptotically stationary is that $1-\alpha<k<\alpha-1$. Therefore, we summarized the above results by the following theorem 1.

Theorem 1: The Full Range Auto Regressive (FRAR) process $\{X_t\}$ is asymptotically stationary and identifiable if and only if the domain of the parameter space S is $\{k, \alpha, \theta, \phi / k\in R,\ 1-\alpha<k<\alpha-1,\ \in[0,\pi),\ \phi\in[0,\pi/2)\}$ $\alpha>1$.

Thus, the new FRAR model incorporates long range dependence, involves only four parameters and is totally free from order determination problems.

Bayesian analysis of frar model

The posterior analysis

The Bayesian approach to the analyses of the new model consists in determining the posterior distribution of the parameters of the FRAR model and the predictive distribution of future observations. From the former, one makes posterior inferences about the parameters of the FRAR model including the variance of the white noise. From the latter, one may forecast future observations. All these techniques are illustrated by Broemeling[4] for autoregressive models.

We shall consider the FRAR model and assume that it is asymptotically stationary and identifiable.

The problem is to estimate the unknown parameters k, α, θ, ϕ and σ^2, using the Bayesian methodology on the basis of a past random realization of $\{X_t\}$ say $x=(x_1,x_2,\dots,x_N)$.

The joint probability density of $X_1,X_2,\dots,X_N$ is given by

$$P(X/\Theta)\propto\left(\sigma^2\right)^{-N/2}\exp\left[-\frac{1}{2\sigma^2}\sum_{t=1}^{2}\left(x_t-k\sum_{r=1}^{\infty}a_r\,x_{t-r}\right)^2\right] \quad (6)$$

where $x=(x_1,x_2,\dots,x_N)$, $\Theta=\left(k,\alpha,\theta,\varphi,\sigma^2\right)$ and $a_r=\left(1/\alpha^2\right)\sin(r\theta)\cos(r\phi)$.

The notation P is used as a general notation for the probability density function of the random variables given within the parentheses following P and $X_0,X_{-1},X_{-2},\dots$ are the past realizations on X_t which are unknown. Following Priestley [2] and Broemeling [3], these are assumed to be zero for the purpose of deriving the posterior distribution of Θ. Therefore, the range for the index r, viz., 1 through ∞, reduces to 1 through N and so, in the joint probability density function of the observations given by (6), the range of the summation 1 through ∞ can be replaced by 1 through N. By expanding the square in the exponent and simplifying, one gets

$$P(X/\Theta)\propto\left(\sigma^2\right)^{-N/2}\exp\left(-Q/2\sigma^2\right) \quad (7)$$

where $Q=T_{00}+k^2\sum_{r=1}^{N}a_r^2T_{rr}+2k^2\sum_{r<s;r,s=1}^{N}a_ra_sT_{rs}-2k\sum_{r=1}^{N}a_rT_{r0}$, $T_{rs}=\sum_{t=1}^{N}x_{t-r}x_{t-s}$, $r,s=0,1,\dots,N$, $\Theta\in S$.

To find the posterior distribution of Θ we first have to specify the prior distribution for the parameters.

α is distributed as the displaced exponential distribution(since it is bigger than 1) with parameter β; σ^2 has the inverted gamma distribution with parameter v and δ; k, θ and ϕ are uniformly

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distributed over their domain.

Thus, the joint prior density function of Θ $(\Theta \in S)$ is given by

$$P(\Theta) \propto \beta \exp\left(-\beta(\alpha-1)-\nu/\sigma^2\right)\left(\sigma^2\right)^{-(\delta+1)} \quad (8)$$

Using (7), (8), and Bayes' theorem, the joint posterior density of k, α, θ, ϕ and σ^2 is obtained as

$$P(\Theta/X) \propto \left(\sigma^2\right)^{-N/2} \exp\left(-Q/2\sigma^2\right) \exp\left[-\beta(\alpha-1)-\nu/\sigma^2\right] \left(\sigma^2\right)^{-(\delta+1)} \quad (9)$$

$$\propto \exp\left[-\beta(\alpha-1)\right]\exp\left[-1/2\sigma^2(Q+2\nu)\right] \left(\sigma^2\right)^{-[(N/2)+\delta+1]} \quad (10)$$

Integrating σ^2 out of this joint posterior distribution, we obtain the joint posterior distribution of k, α, θ and ϕ,

$$P(k,\alpha,\theta,\phi/X) \propto e^{-\beta(\alpha-1)}\left\{C\left[1+A_1\left(k-B_1\right)^2\right]\right\}^{-d} \quad (11)$$

where $C=T_{00}-B^2/A+2\nu$; $B=\sum_{r=1}^{N} a_r T_{0r}$;

$A=\sum_{r=1}^{N} a_r^2 T_{rr}+2\sum_{r,s=1}^{N} a_r a_s T_{rs}$; $A_1=A/C$; $B_1=B/C$; $d=\dfrac{N}{2}+\delta$.

Thus, the posterior distribution of k conditional on α, θ and ϕ is a *t*-distribution located at B_1 with $(2d-1)$ degrees of freedom.

Thus, the joint posterior density function of α, θ and ϕ can be obtained by integrating with respect to *k*. Thus,

$$P(\alpha,\theta,\phi/x) \propto \exp\left(-\beta(\alpha-1)\right)C^{-d} A_1^{-1/2}; \text{ with } \alpha>1,\ 0\le\theta<\pi \text{ and } 0\le\phi<\pi/2. \quad (12)$$

The above joint posterior density of α, θ and ϕ is a very complicated expression and is analytically intractable. One way of solving the problem is to find the marginal posterior density of α, θ and ϕ from the joint density (12) using ordinary numerical integration, using FORTRAN.

One-step-ahead prediction

In order to forecast x_{N+1} using the random realization $x_1,x_2,\ldots,x_N$ on $(X_1,X_2,\ldots,X_N)$, one must find the conditional distribution of X_{N+1} given the past observations. This is the predictive distribution of X_{N+1} and will be derived by multiplying the conditional density of X_{N+1} given $X_1,X_2,\ldots,X_N$, Θ and the posterior density of Θ given $X_1,X_2,\ldots,X_N$ and then integrating with respect to Θ. That is,

$$P\left(X_{N+1}/X_1,X_2,\ldots,X_N\right)=\int_\Theta P\left(X_{N+1}/X_1,X_2,\ldots,X_N,\Theta\right)P\left(\Theta/X_1,X_2,\ldots,X_N\right)d\Theta \cdot$$

Thus, we obtain

$$P\left(x_{N+1}/x_1,x_2,\ldots,x_N,\Theta\right) \propto \left(\sigma^2\right)^{-1/2} \exp\left[-\frac{1}{2\sigma^2}\left(x_{N+1}-k\sum_{i=1}^{\infty} a_i\, x_{N+1-i}\right)^2\right],$$

$$x_{N+1}\in R. \quad (13)$$

The square in the exponent in the above expression, say Q_1, can be rewritten, after expanding the square, as $Q_1=x_{N+1}^2+k^2\sum_{i=1}^{N} a_i^2 P_i^2+2k^2\sum_{i<j;i=1}^{N} a_i\, a_j\, P_{ij} -2k\sum_{i=1}^{N} a_i\, P_i\, X_{N+1}$, where $P_i=X_{N+1-i}$ and $P_{ij}=X_{N+1-i}X_{N+1-j}$. Now multiplying (13) by the joint posterior density of Θ and integrating over the parameter space Θ, we obtain,

$$P\left(x_{N+1}/x_1,x_2,\ldots,x_N\right) \propto \int \exp\left(-\beta(\alpha-1)\right)\left(1/\sigma^2\right)^{\left[\frac{N}{2}+\delta+\frac{1}{2}+1\right]} \exp\left[-\frac{1}{2\sigma^2}\left(Q+Q_1+2\upsilon\right)\right]d\Theta \quad (14)$$

First, integrating out σ^2 in (14), one gets the joint distribution of x_{N+1}, k, α, θ and ϕ as

$$P\left(x_{N+1},k,\alpha,\theta,\phi/x_1,x_2,\ldots,x_N\right) \propto \exp\left(-\beta(\alpha-1)\right)\left(Q+Q_1+2\upsilon\right)^{-\left(\frac{N+1}{2}+\delta\right)} \quad (15)$$

where $d_1=\sum_{i=1}^{N} a_i^2 T_{ii}+2\sum_{i<j;i=1}^{N} a_i a_j T_{ij}$, $d_2=\sum_{i=1}^{N} a_i^2 P_i^2+2\sum_{i<j;i=1}^{N} a_i a_j P_{ij}$, $d_3=\sum_{i=1}^{N} a_i T_{i0}$, $d_4=\sum_{i=1}^{N} a_i P_i$;

$$\left(Q+Q_1+2\upsilon\right)=k^2\left(d_1+d_2\right)-2k\left(d_3+d_4 x_{N+1}\right)+\left(x_{N+1}^2+T_{00}+2\upsilon\right).$$

Thus,

$$P\left(x_{N+1},k,\alpha,\theta,\phi/x_1,x_2,\ldots,x_n\right) \propto \exp\left(-\beta(\alpha-1)\right)C_1\left[1+E_1\left(k-C_2\right)^2\right]^{-d} \quad (16)$$

where $C_1=\left\{x_{N+1}^2+T_{00}+2\upsilon-\left[\left(d_3+d_4 x_{N+1}\right)^2/\left(d_1+d_2\right)\right]\right\}$, $C_2=\left(d_3+d_4 x_{N+1}\right)/\left(d_1+d_2\right)$, $E_1=\left(d_1+d_2\right)/C_1$.

Further, integrating out k from (16) we get

$$P\left(x_{N+1},k,\alpha,\theta,\phi/x_1,x_2,\ldots,x_N\right) \propto \exp\left(-\beta(\alpha-1)\right)C_1^{-d} E_1^{-(1/2)} \quad (17)$$

with $d=(\upsilon+1)/2$ which is the conditional predictive distribution of x_{N+1} given α, θ and ϕ. Further elimination of the parameters α, θ and ϕ from (17) is not possible analytically. So the marginal posterior density of x_{N+1} cannot be expressed in a closed form. Since the distribution in (17) is analytically not tractable, a complete Bayesian analysis is possible by numerical integration technique or simulation based approach, viz., MCMC technique.

Suppose one wants a point estimate (posterior mean) of , then one should compute the marginal posterior density of x_{N+1} from (17) and use it to calculate the marginal posterior mean of x_{N+1}. Thus four dimensional numerical integration is necessary in order to estimate x_{N+1}. But it is a very difficult problem.

Practically, to perform four dimensional numerical integration is very difficult and therefore to reduce the dimensions of the numerical integration one may substitute the estimators, posterior means, $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\phi}$ respectively in the place of α, θ and ϕ and then perform one dimensional numerical integration to find the conditional mean of X_{N+1}. That is, one may eliminate the parameters as much as possible by analytical methods and then use the conditional estimates for the remaining parameters to compute the marginal posterior mean of the future observation.

Numerical example - canadian lynx data

A numerical example is considered for illustrating the one-step ahead predictive analysis of a future observation from the Canadian Lynx data. This data consists of the annual record of the numbers of Canadian Lynx trapped in the Mackenzie River district of North-West Canada for the period 1821 – 1934 (both years inclusive) giving a total of 114 observations. Brockwel and Davis[5] (page 501) have transformed these data using the log transformation for the purpose of statistical analysis. These transformed data are used in our Bayesian predictive analysis.

Bayesian predictive distribution of the $(r+1)^{th}$ observation, using the r observation, is obtained. The mean of this distribution is taken to be the $(r+1)^{th}$ predicted value of the Lynx data. Since the direct evaluation of the mean of the one-step ahead predictive distribution

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involves four dimensional numerical integration, instead of the marginal predictive distribution of X_{N+1}, the conditional predictive distribution of X_{N+1}, given by (17) got by fixing the parameters k, α, θ and ϕ at their estimates, is used and the mean (posterior mean) is calculated using FORTRAN language. The posterior mean of the predictive distribution is computed numerically after fixing the parameters at their respective estimated values. The prediction is done for the cases r=11, 12,….,114 by taking first 10 observations as initial observations to estimate the parameters of the model and are given in the Table 1 which contains both the true values and the one-step ahead predicted values for the transformed data and the figure 1 represent graphically, the original data and one step-step ahead predicted values of the same. Figure 2 represent graphically, the original data for the last 14 observations and predicted values of the same through different methods, using FORTRAN program.

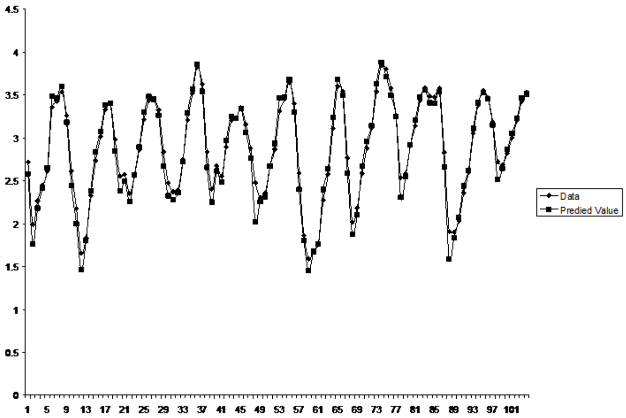

Figure 1 Original data and one step-step ahead predicted values of the same.

A comparison of the one-step ahead predicted values using FRAR model with other models relating to this data available in the literatures are discussed in the following Section.

Comparative study

Lin[6] has studied the Canadian lynx data through various time series models and Nicholls and Quinn[7] have used the Canadian lynx data to compare the quality of the predicted values obtained by several methods, viz., (1) Moran-1 (2) Tong (3) NQ-1 (4) Moran-2 and (5) NQ-2 as presented above.

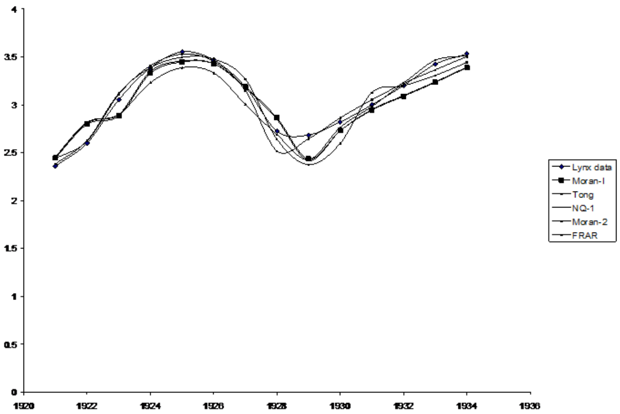

Figure 2 Predicted values through different methods.

Moran-I refers to the linear predictor obtained from the second order autoregressive model, Tong refers to the linear predictor from autoregressive model of order eleven, NQ-1 denotes the linear predictor obtained from the second order random coefficient model while Moran-2 and NQ-2 denotes the non-linear predictors for the lynx data. The models and other details can found in the Nicholls and Quinn.[7]

Nicholls and Quinn[7] have used these methods to predict the last 14 values of the Canadian lynx data and calculated the error sum of squares (refer Table 8.1 in page 146). To compare the efficiency of prediction of the new FRAR model developed in this paper with those of the others stated above, the Table cited above is reproduced in Table 2 wherein the values predicted by the FRAR model are given as an additional column. The error sum of squares for the last 14 predicted values is 0.0637 under the FRAR model whereas they are 0.2531, 0.2541, 0.2561, 0.2070 and 0.1887 respectively under the other methods. So, at least in the above context the superiority of the FRAR model is established beyond doubt.

Summary and conclusion

The Full Range Autoregressive model provides an acceptable alternative to the existing methodology. The main advantage associated with the new method is that one is completely avoiding the problem of order determination of the model as in the existing methods.

Table 1 One-Step-ahead predicted values of the transformed Lynx data

S. No.	Y	$\hat{Y}$	S. No.	Y	$\hat{Y}$	S. No.	Y	$\hat{Y}$
1	2.430	-	41	2.373	2.283	81	2.880	2.963
2	2.506	-	42	2.389	2.360	82	3.115	3.143
3	2.767	-	43	2.742	2.726	83	3.540	3.633
4	2.940	-	44	3.210	3.292	84	3.845	3.881
5	3.169	-	45	3.520	3.569	85	3.800	3.713
6	3.450	-	46	3.828	3.856	86	3.579	3.494
7	3.594	-	47	3.628	3.542	87	3.264	3.249
8	3.774	-	48	2.837	2.656	88	2.538	2.306
9	3.695	-	49	2.406	2.252	89	2.582	2.547
10	3.411	-	50	2.675	2.614	90	2.907	2.917

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Table Continued

S. No.	Y	$\hat{Y}$	S. No.	Y	$\hat{Y}$	S. No.	Y	$\hat{Y}$
11	2.718	2.582	51	2.554	2.481	91	3.142	3.204
12	1.991	1.767	52	2.894	2.973	92	3.433	3.473
13	2.265	2.181	53	3.202	3.248	93	3.580	3.562
14	2.446	2.413	54	3.224	3.229	94	3.490	3.408
15	2.612	2.650	55	3.352	3.344	95	3.475	3.406
16	3.359	3.482	56	3.154	3.062	96	3.579	3.539
17	3.429	3.468	57	2.878	2.765	97	2.829	2.663
18	3.533	3.596	58	2.476	2.023	98	1.909	1.587
19	3.261	3.182	59	2.303	2.255	99	1.903	1.833
20	2.612	2.444	60	2.360	2.315	100	2.033	2.069
21	2.179	1.999	61	2.671	2.672	101	2.360	2.439
22	1.653	1.461	62	2.867	2.934	102	2.601	2.621
23	1.832	1.801	63	3.310	3.466	103	3.054	3.108
24	2.328	2.385	64	3.449	3.479	104	3.386	3.409
25	2.737	2.839	65	3.646	3.684	105	3.553	3.528
26	3.014	3.069	66	3.400	3.296	106	3.468	3.454
27	3.328	3.380	67	2.590	2.399	107	3.187	3.150
28	3.404	3.405	68	1.863	1.806	108	2.723	2.518
29	2.981	2.849	69	1.591	1.454	109	2.686	2.646
30	2.557	2.379	70	1.690	1.677	110	2.821	2.864
31	2.576	2.500	71	1.771	1.766	111	3.000	3.053
32	2.352	2.260	72	2.274	2.398	112	3.201	3.231
33	2.556	2.569	73	2.576	2.642	113	3.424	3.464
34	2.864	2.895	74	3.111	3.241	114	3.531	3.512
35	3.214	3.296	75	3.605	3.683			
36	3.435	3.481	76	3.543	3.499			
37	3.458	3.449	77	2.769	2.589			
38	3.326	3.263	78	2.021	1.877			
39	2.835	2.668	79	2.185	2.105			
40	2.476	2.325	80	2.588	2.671			

Y – Lynx (Transformed)
- one-step-ahead
Predicted value

Table 2 One-Step ahead predictors of the transformed lynx data.

S.No	Year	Lynx data	Moran-1	Tong	NQ-1	Moran-2	NQ-2	FRAR
1	1921	2.3598	2.4448	2.4559	2.4596	2.3835	2.3842	2.4390
2	1922	2.6010	2.7971	2.8088	2.8173	2.6271	2.6323	2.6210
3	1923	3.0538	2.8850	2.8991	2.8989	3.1193	3.0955	3.1080
4	1924	3.3860	3.3285	3.2306	3.3474	3.3883	3.3971	3.4090
5	1925	3.5532	3.4471	3.3879	3.4571	3.4955	3.4999	3.5280
6	1926	3.4676	3.4289	3.3321	3.4296	3.4787	3.4781	3.4540
7	1927	3.1867	3.1859	3.0060	3.1759	3.2683	3.2555	3.1500
8	1928	2.7235	2.8628	2.6875	2.8468	2.6405	2.6587	2.5180
9	1929	2.6857	2.4348	2.4286	2.4153	2.3747	2.3650	2.6460

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Table Continued

S.No	Year	Lynx data	Moran-I	Tong	NQ-I	Moran-2	NQ-2	FRAR
10	1930	2.8209	2.7296	2.7643	2.7299	2.5977	2.6292	2.8640
11	1931	3.0000	2.9440	2.9838	2.9508	3.1277	3.0927	3.0530
12	1932	3.2014	3.0897	3.2169	3.0966	3.1981	3.1762	3.2310
13	1933	3.4244	3.2331	3.3656	3.2390	3.3065	3.2956	3.4640
14	1934	3.5309	3.3896	3.5035	3.3942	3.443	3.4413	3.5120
Error sum of squares			0.2531	0.2541	0.2561	0.2070	0.1887	0.0637

Thus, it is not unreasonable to claim the FRAR model proposed and its Bayesian analysis presented above certainly provides a viable alternative to the existing time series methodology, completely avoiding the problem of order determination.

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Conflicts of interest

None

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