Research Article

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Length and area biased exponentiated weibull distribution based on forest inventories

Abstract

Length-biased and Area-biased distributions arise in many forestry applications, as well as other environmental, econometric, and biomedical sampling problems. We examine the Length-biased and Area-biased distributions versions of the Exponentiated Weibull distribution (EW). This study introduced a new distribution based on Length-biased Exponentiated Weibull distribution (LBEW) and Area-biased Exponentiated Weibull distribution (ABEW). Some characteristics of the new distributions were obtained. Plots for the cumulative distribution function, pdf and tables with values of skewness and kurtosis were also provided. Height-Diameter (H-D) data on Bombax and Pines (*Pinus caribaea*) were used to demonstrate the application of the distributions. Estimation of parameters of EW, LBEW and ABEW distributions were done using the maximum likelihood approach and compared across the distributions using criteria like AIC and Loglikelihood. We therefore proposed that similar to Exponentiated Weibull distribution (EW), a better fitting of Bombax and Pines H-D data are possible by LBEW and ABEW distributions. We hope in numerous fields of theoretical and applied sciences, the findings of this study will be useful for the practitioners.

Keywords: length-biased distribution, area-biased distributions, forestry, bombax, pines, height and diameter

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Introduction

Trees contribute to the environment by providing oxygen, improving air quality, climate amelioration, conserving water, preserving soil, and supporting wildlife. During the process of photosynthesis, trees take in carbon dioxide and produce the oxygen we breathe. According to the U.S. Department of Agriculture, "One acre of forest absorbs six tons of carbon dioxide and puts out four tons of oxygen. This is enough to meet the annual needs of 18 people." Trees, shrubs and turf also filter air by removing dust and absorbing other pollutants like carbon monoxide, sulphur dioxide and nitrogen dioxide. After trees intercept unhealthy particles, rain washes them to the ground. Trees can add value to your home, help cool your home and neighborhood break the cold winds to lower your heating costs, and provide food for wildlife.

Height-diameter relationships are used to estimate the heights of trees measured for their diameter at breast height (DBH). Such relationship describes the correlation between height and diameter of the trees in a stand on a given date and can be represented by a linear or non-linear statistical model. In forest inventory designs diameter at breast height is measured for all trees within sample plots, while height is measured for only some selected trees, normally the dominant ones in terms of their DBH. In this study, the two species of trees considered explained thus;

i. ***Pinus caribaea:*** *'Pinus'* is from the Greek word 'pinos' (pine tree), possibly from the Celtic term 'pin' or 'pyn' (mountain or rock), referring to the habitat of the pine. *Pinus caribaea* is a fine tree to 20-30m tall, often 35m, with a diameter of 50-80cm and occasionally up to 1m; trunk generally straight and well formed; lower branches large, horizontal and drooping; upper branches often ascending to form an open, rounded to pyramidal crown; young trees with a dense, pyramidal crown. *Pinus caribaea* is rated as moderately fire resistant. It tolerates salt winds and hence may be planted near the coast.

ii. ***Bombax costatum:*** *'Bombax'* is derived from the Greek '*bombux*', meaning silk, alluding to the dense wool-like floss covering the inner walls of the fruits and the seeds. *Bombax costatum* is a fire resisting tree of the savannas and dry woodlands from Senegal to central Africa, from Guinea across Ghana and Nigeria to southern Chad. Its tuberous roots act as water and/or sugar storage facilities during long drought periods. Usually associated with *Pterocarpus erinaceus, Daniellia oliveri, Cordyla pinnata, Parkia biglobosa, Terminalia macroptera and Prosopis africana*.

Length-biased and area-biased distribution

When the probability of selecting an individual in a population is proportional to its magnitude, it is called length biased sampling. However, when observations are selected with probability proportional to their length, the resulting distribution is called length-biased. When dealing with the problem of sampling and selection from a length-biased distribution, the possible bias due to the nature of data-collection process can be utilized to connect the population parameters to that of the sampling distribution. That is, biased sampling is not always detrimental to the process of inference on population parameters. Inference based on a biased sample of a certain size may yield more information than that given by an unbiased sample of the same size, provided that the choice mechanism behind the biased sample is known. Statistical analysis based on length-biased samples has been studied in detail since the early 70's. Size-biased distributions have been found to be useful in probability sampling designs for forestry and other related studies. These designs are classified into length-biased methods where sampling is done with probability proportional to some lineal measure and area-biased methods where units are selected into the sample with probability proportional to some real attributes. Hence, area-biased distribution is the square of the random variable of X or the second order power of size-biased distribution

The concept of length-biased was introduced by Cox in 1962.[1] This concept is found in various applications in biomedical area such as family history and disease, survival analysis, intermediate events and latency period of AIDS due to blood transfusion. Many works were done to characterize relationships between original distributions and their length-biased versions. Patill and Rao expressed some basic distributions and their length-biased forms such as log-normal, gamma, pareto, beta distributions. Recently, many researches are applied to length-biased for lifetime distribution, length-biased weighted Weibull distribution, and length-biased weighted generalized Rayleigh distribution, length-biased beta distribution, and Bayes estimation of length- biased Weibull distribution.[2]

Exponentiated weibull distribution

The Weibull distribution was introduced by Wallodi Weibull, Swedish scientist, in 1951. It is perhaps the most widely used distribution to analyze the lifetime data. Gupta & Kundu[3] proposed an Exponentiated Exponential distribution which is a special case of the Exponentiated Weibull family. Flaih et al.,[4] extended the Inverted Weibull distribution to the Exponentiated Inverted Weibull (EIW) distribution by adding another shape parameter. This study suggested that the EIW distribution can provide a better fit to the real dataset than the IW distribution. Shittu, O I. and Adepoju, K A.[5] the exponentiated Weibull was used as an alternative distribution that adequately describe the wind speed and thereby provide better representation of the potentials of wind energy.

Structural properties of exponentiated weibull distribution: According to Mudhokar, et al.,[6] the Exponentiated Weibull density function is defined as;

$$f_{EW}(x;k,\lambda,\alpha)=\frac{\alpha k x^{k-1}}{\lambda^{k}}\exp\left(-(x/\lambda)^{k}\right)\left(1-\exp\left(-(x/\lambda)^{k}\right)\right)^{\alpha-1} \quad (1)$$

and the cdf is;

$$F_{EW}(x;k,\lambda,\alpha)=\left(1-\exp\left(-(x/\lambda)^{k}\right)\right)^{\alpha}$$

α and k are shape parameters; λ is a scale parameter.

the rth moment of the exponentiated weibull is given as;

$$E(x^{r})=\alpha\lambda^{r}\Gamma\left(\frac{r}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{r}{k}+1}}$$

Where $\Gamma\left(\frac{r}{k}+1\right)=\int_{0}^{\infty}x^{\frac{r}{k}}\exp(-x)\partial x$ at r=1, the **first moment** of EW is

$$E(x)=\alpha\lambda\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}$$

- at r=2 is the second moment and the **variance** of EW is given thus;

$$\mathrm{Var(x)}=E(x^{2})-[E(x)]^{2}$$

$$\mathrm{Var(x)}=\alpha\lambda^{2}\Gamma\left(\frac{2}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{2}{k}+1}}-\alpha^{2}\lambda^{2}\Gamma^{2}\left(\frac{1}{k}+1\right)\left[\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}\right]^{2}$$

The **skewness** and **kurtosis** of EW

$$k_{3}=\frac{\alpha\lambda\Gamma\left(\frac{3}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{3}{k}+1}}-\left[3\alpha\lambda\Gamma\left(\frac{2}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{2}{k}+1}}\right]\left[\alpha\lambda\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}\right]+2\alpha\lambda\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}{\left[\mathrm{Var(x)}\right]^{3/2}}$$

Materials and methods

In this study, we propose two new distributions which are LBEW and ABEW distributions. We first provide a general definition of the Length-biased and Area-biased distributions which we subsequently reveal their pdfs.

Let $f(x;\theta)$ be the pdf of the random variable X and θ be the unknown parameter. The weighted distribution is defined as;

$$k_{4}=\frac{\alpha\lambda\Gamma\left(\frac{4}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{4}{k}+1}}-\left[4\alpha\lambda\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}\right]\left[\alpha\lambda\Gamma\left(\frac{3}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{3}{k}+1}}\right]+\left[6\alpha^{2}\lambda\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{2}{k}+1}}\right]\left[\alpha\lambda\Gamma\left(\frac{2}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{2}{k}+1}}\right]-3\alpha^{4}\lambda\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}{\left[\mathrm{Var(x)}\right]^{2}}$$

$$g(x;\theta)=\frac{x^m f(x;\theta)}{E[f(x)]}\quad X\in R,\ \theta>0 \qquad (2)$$

The distributions in equation (2) are termed as size-biased distribution of order m. When m=1, it is called size-biased of order 1 or say length biased distribution, whereas for m=2, it is called the area- biased distribution.

Length-biased EW distribution (LBEW)

If X has a lifetime distribution with pdf $f(x)$ and expected value, $E\left[f(x)\right]<\infty$, the pdf of length-biased distribution of X can be defined as:

$$g_{LB}(x;\theta)=\frac{xf(x;\theta)}{E[f(x)]} \qquad (3)$$

Let X be a random variable of an EW distribution with pdf $f(x)$.

Then $g_{LB}(x;\theta)=\frac{xf(x;\theta)}{E[f(x)]}$ is a pdf of the LBEW distribution with two shape parameters α and k and a scale parameter λ. The notation for X with the LBEW distribution is denoted as X ~LBEW (α , k , λ). The pdf of X is given by:

$$g_{LBEW}(x;k,\lambda,\alpha)=\frac{\frac{kx^k}{\lambda^{k+1}}\exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}} \qquad (4)$$

Area-biased EW distribution (ABEW)

If X has a lifetime distribution with pdf $f(x)$ and expected value, $E\left[f(X)\right]<\infty$, the pdf of length-biased distribution of X can be defined as:

$$g_{AB}(x;\theta)=\frac{x^2 f(x;\theta)}{E[f(x)]} \qquad (5)$$

Let X be a random variable of an EW distribution with pdf $f(x)$. Then $g_{AB}(x;\theta)=\frac{x^2 f(x;\theta)}{E[f(x)]}$ is a pdf of the ABEW distribution with two shape parameters α and k and a scale parameter λ. The notation for X with the ABEW distribution is denoted as $X\sim ABEW$ (α , k , λ). The pdf of X is given by:

$$g_{ABEW}(x;k,\lambda,\alpha)=\frac{\frac{kx^{k+1}}{\lambda^{k+1}}\exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}} \qquad (6)$$

The properties

The LBEW distribution properties are as follows;

$$\int_0^\infty g_{LBEW}(x)\partial x=\int_0^\infty\frac{\frac{kx^k}{\lambda^{k+1}}\exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\partial x$$

$$=\frac{\frac{k}{\lambda^{k+1}}}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\int_0^\infty x^k\exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}\partial x$$

If $y=(x/\lambda)^k$, then we have that;

$$=\frac{1}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\int_0^\infty y^{\frac{1}{k}+2}\exp(-y)\left(1-\exp(-y)\right)^{\alpha-1}\partial y$$

Recall that,

$$\left(1-\exp(-y)\right)^{\alpha-1}=\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\exp(-yj)$$

Therefore,

$$\int_0^\infty g_{LBEW}(x)\partial x=\frac{1}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\int_0^\infty y^{\frac{1}{k}+2}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\exp\left(-y[1+j]\right)\partial y$$

; where $m=y\left[1+j\right]$

$$=\frac{1}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\left[\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}\right]\int_0^\infty m^{\frac{1}{k}+2}\exp(-m)\partial m$$

$$\int_0^\infty g_{LBEW}(x)\partial x=\frac{1}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\left[\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}\right]\Gamma\left(\frac{1}{k}+1\right)$$

$$\int_0^\infty g_{LBEW}(x)\partial x=1$$

Therefore, the pdf of LBEW distribution sum to 1. ***NB:*** It was also obtainable for the ABEW distribution.

The **cdf** of LBEW, corresponding to (4) is obtained by

$$F_{LBEW}(X) = \int_0^x \frac{\frac{kx^k}{\lambda^{k+1}} \exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\partial x$$

$$F_{LBEW}(X) = \frac{\frac{k}{\lambda^{k+1}}}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\int_0^x x^k \exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}\partial x$$

Let $y=(x/\lambda)^k;\, y\lambda^k = x^k;\, x=\lambda y^{1/k}$

$$F_{LBEW}(X) = \frac{1}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\int_0^x y^k \exp\left(-y^k\right)\left(1-\exp\left(-y^k\right)\right)^{\alpha-1}\partial x$$

Let $m = y(1+j);\, y=\frac{m}{1+j}$

$$F_{LBEW}(X) = \frac{1}{\Gamma\left(\frac{1}{k}+1\right)}\int_0^x m^{1/k}\exp(-m)\partial m$$

$$F_{LBEW}(X) = \frac{x^{1/k}\exp(-x)\left[1+\frac{1}{kx}\right]}{\Gamma\left(\frac{1}{k}+1\right)\left[1-\frac{1}{k}+\frac{1}{k^2}\right]} \quad \text{.................................. (7)}$$

So, the **reliability function** of LBEW is,

$$R(x) = 1-F_{LBEW}(x) = 1-\frac{x^{1/k}\exp(-x)\left[1+\frac{1}{kx}\right]}{\Gamma\left(\frac{1}{k}+1\right)\left[1-\frac{1}{k}+\frac{1}{k^2}\right]} \quad \text{..... .... (8)}$$

$$R(x) = \frac{\Gamma\left(\frac{1}{k}+1\right)\left[1-\frac{1}{k}+\frac{1}{k^2}\right]-\left[1+\frac{1}{kx}\right]x^{1/k}\exp(-x)}{\Gamma\left(\frac{1}{k}+1\right)\left[1-\frac{1}{k}+\frac{1}{k^2}\right]}$$

And the **hazard function** is,

$$h(x) = \frac{f(x)}{R(x)} = \frac{\frac{kx^k}{\lambda^{k+1}}\exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}\left[1-\frac{1}{k}+\frac{1}{k^2}\right]}{\left\{\Gamma\left(\frac{1}{k}+1\right)\left[1-\frac{1}{k}+\frac{1}{k^2}\right]-\left[1+\frac{1}{kx}\right]x^{1/k}\exp(-x)\right\}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}} \quad \text{........(9)}$$

The moments

The ***r*th raw moment** of the LBEW random variable X is

$$E(X^r) = \frac{1}{\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}\int_0^{\infty} x^r\frac{\alpha kx^k}{\lambda^{k+1}}\exp\left(-(x/\lambda)^k\right)\left(1-\exp\left(-(x/\lambda)^k\right)\right)^{\alpha-1}\partial x$$

$$E(X^r) = \frac{1}{\lambda^r\Gamma\left(\frac{1}{k}+1\right)}\Gamma\left(\frac{r+1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(1+j)^{(1/k)+1}}{(1+j)^{((r+1)/k)+1}} \quad \text{..................... (10)}$$

at r = 1, the **first moment** of LBEW is

$$E(X) = \frac{1}{\lambda\Gamma\left(\frac{1}{k}+1\right)}\Gamma\left(\frac{2}{k}+1\right)\sum_{j=0}^{\infty}(1+j)^{-(1/k)} \quad \text{........ (11)}$$

And the **variance** is

$$Var(X) = \frac{1}{\lambda^2\Gamma\left(\frac{1}{k}+1\right)}\Gamma\left(\frac{3}{k}+1\right)\sum_{j=0}^{\infty}(1+j)^{-(1/k)} - \left[\frac{1}{\lambda\Gamma\left(\frac{1}{k}+1\right)}\Gamma\left(\frac{2}{k}+1\right)\sum_{j=0}^{\infty}(1+j)^{-(1/k)}\right]^2 \quad \text{............. (12)}$$

The **skewness** and **kurtosis** of LBEW;

$$k_3 = \frac{\alpha\lambda\Gamma\left(\frac{3}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{3}{k}+1}}-\left[6\alpha^2\lambda\Gamma\left(\frac{4}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{4}{k}+1}}\right]\left[\alpha^2\lambda\Gamma\left(\frac{2}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{2}{k}+1}}\right]+8\alpha^2\lambda^2\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{1}{k}+1}}}{\left[Var(X)\right]^{3/2}} \quad \text{............(13)}$$

$$k_4 = \frac{\alpha\lambda^2\Gamma\left(\frac{4}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{4}{k}+1}}-\left[8\alpha^2\lambda^2\Gamma\left(\frac{3}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{3}{k}+1}}\right]\left[\alpha^2\lambda\Gamma\left(\frac{4}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{4}{k}+1}}\right]+\left[6\alpha^2\lambda^2\Gamma\left(\frac{2}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)^{\frac{2}{k}+1}}\right]}{\left[Var(X)\right]^2} \quad \text{...........(14)}$$

From equation (7), (8), (9), (10), (11), (12), (13), (14), we established all the properties of LBEW distribution and also that of ABEW was obtained which can be fetch in the body of the work.

Maximum likelihood approach

Harter and Moore (1965) were the earliest statisticians to use the maximum likelihood procedure because of its desirable characteristics.

The three distributions in the study (EW, LBEW and ABEW) are solved iteratively by computer algorithm to obtain the maximum likelihood estimates of the parameters α , k and λ .

MLE of EW

Let X be a random sample of size *n* from the EW distribution given by equation (1). Then the log likelihood function comes out to be

$$L(\alpha,\lambda,k)=n\ln\alpha+n\ln k+nk\ln\lambda+(k-1)\sum\ln x_i+$$
$$(\alpha-1)\sum\ln\left[1-\exp\{-(x_i/\lambda)^k\}\right]-\sum(x_i/\lambda)^k \quad \ldots\ldots\ldots\ldots (15)$$

Therefore the MLEs of α , λ , k which maximize (15) must satisfy the normal equations given by

Derivative w.r.t α

$$\frac{\partial}{\partial\alpha}L(\alpha,\lambda,k)=\frac{n}{\alpha}+\sum\ln\left[1-\exp\{-(x_i/\lambda)^k\}\right]=0$$

We obtain the MLE of α as

$$\hat{\alpha}=-\frac{n}{\sum\ln\left[1-\exp\{-(x_i/\lambda)^k\}\right]} \quad \ldots\ldots\ldots\ldots (16)$$

Derivative w.r.t λ

$$\frac{\partial}{\partial\lambda}L(\alpha,\lambda,k)=\frac{nk}{\lambda}+(\alpha-1)k\lambda^{k-1}\sum\frac{\exp\{-(x_i/\lambda)^k\}}{1-\exp\{-(x_i/\lambda)^k\}}x_i-k\lambda^{k-1}\sum x_i^k=0$$

Multiplying the above equation by $\frac{\lambda}{k}$ we get

$$n+\lambda^k\left[(\alpha-1)\sum\frac{\exp\{-(x_i/\lambda)^k\}}{1-\exp\{-(x_i/\lambda)^k\}}x_i^k-\sum x_i^k\right]=0 \quad \ldots\ldots (17)$$

Derivative w.r.t k

$$\frac{\partial}{\partial k}L(\alpha,\lambda,k)=\frac{n}{k}+n\ln\lambda+\sum\ln x_i+(\alpha-1)\lambda^k\sum\frac{\exp\{-(x_i/\lambda)^k\}}{1-\exp\{-(x_i/\lambda)^k\}}x_i^k\ln(x_i/\lambda)-\lambda^{k-1}\sum x_i^k\ln(x_i/\lambda)=0$$

then,

$$\frac{n}{k}+\sum\ln x_i+\lambda^k\left[(\alpha-1)\sum\frac{\exp\{-(x_i/\lambda)^k\}}{1-\exp\{-(x_i/\lambda)^k\}}x_i^k\ln x_i-\sum x_i^k\ln x_i\right]=0 \quad \ldots\ldots\ldots (18)$$

Using (15) in (17) and (18) we get equations, which are satisfied by the MLEs $\hat{\lambda}$ and $\hat{k}$ of λ and k , respectively. Because of the complicated form of the likelihood equations, algebraically it is very difficult to prove that the solution to the normal equations give a global maximum or at least a local maximum, though numerical computation during data analysis showed the presence of at least local maximum.

However, the following properties of the log-likelihood function have been algebraically noted:

for given (λ , k), *log-likelihood* is a strictly concave function of α . Further, the optimal value of α , given by (8), is a concave increasing function of λ , for given k ;

for given (α , k), and $\alpha \geq 1$, *log-likelihood* is a strictly concave function of λ .

MLE of LBEW

Taking the log-likelihood and derivative of the equation (4) to obtain the MLEs of parameters α , k and λ

$$\frac{\partial}{\partial\alpha}L(\alpha,\lambda,k)=\sum_{i=1}^{n}(\alpha-1)\ln\left[1-\exp\{-(x_i/\lambda)^k\}\right]^{\alpha-2}-\frac{\partial}{\partial\alpha}\left\{n\ln\left[\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)\frac{1}{k}+1}\right]\right\}=0 \quad \ldots\ldots\ldots\ldots (19)$$

$$\frac{\partial}{\partial k}L(\alpha,\lambda,k)=\frac{n}{k}+\sum_{i=1}^{n}\frac{k\ln x_i^{k-1}}{x_i^k}-n\ln\lambda-k(x/\lambda)^k-\sum_{i=1}^{n}\left\{\frac{(\alpha-1)\exp\{-(kx_i^k/\lambda^{k+1})\}}{1-\exp\{-(x_i/\lambda)^k\}}\right\}^{\alpha-1}-\frac{\partial}{\partial\alpha}\left\{n\ln\left[\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)\frac{1}{k}+1}\right]\right\}=0 \quad \ldots\ldots\ldots\ldots (20)$$

$$\frac{\partial}{\partial\lambda}L(\alpha,\lambda,k)=-\frac{n(k+1)}{\lambda}+\frac{kx_i^k}{\lambda^{k+1}}-\sum_{i=1}^{n}\left\{\frac{(\alpha-1)\exp\{-(kx_i^k/\lambda^{k+1})\}}{1-\exp\{-(x_i/\lambda)^k\}}\right\}^{\alpha-1}=0 \quad \ldots\ldots\ldots\ldots (21)$$

Equations (19), (20) and (21) are solved iteratively to obtain the maximum likelihood estimates of the parameters α , k and λ .

MLE of ABEW

Taking the log-likelihood and derivative of the equation (6) to obtain the MLEs of parameters α , k and λ

$$\frac{\partial}{\partial\alpha}L(\alpha,\lambda,k)=\sum_{i=1}^{n}(\alpha-1)\ln\left[1-\exp\{-(x_i/\lambda)^k\}\right]^{\alpha-2}-\frac{\partial}{\partial\alpha}\left\{n\ln\left[\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)\frac{1}{k}+1}\right]\right\}=0 \quad \ldots\ldots\ldots\ldots (22)$$

$$\frac{\partial}{\partial k}L(\alpha,\lambda,k)=\frac{n}{k}+\sum_{i=1}^{n}\frac{(k+1)\ln x_i^k}{x_i^{k+1}}-n\ln\lambda-k(x/\lambda)^k-\sum_{i=1}^{n}\left\{\frac{(\alpha-1)\exp\{-(kx_i^k/\lambda^{k+1})\}}{1-\exp\{-(x_i/\lambda)^k\}}\right\}^{\alpha-1}-\frac{\partial}{\partial\alpha}\left\{n\ln\left[\Gamma\left(\frac{1}{k}+1\right)\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}}{(1+j)\frac{1}{k}+1}\right]\right\}=0 \quad \text{.......... (23)}$$

$$\frac{\partial}{\partial\lambda}L(\alpha,\lambda,k)=-\frac{(k+1)}{\lambda}+\frac{kx_i^k}{\lambda^{k+1}}-\sum_{i=1}^{n}\left\{\frac{(\alpha-1)\exp\{-(kx_i^k/\lambda^{k+1})\}}{1-\exp\{-(x_i/\lambda)^k\}}\right\}^{\alpha-1}=0 \quad \text{................ (24)}$$

Also, equations (22), (23) and (24) are solved iteratively to obtain the maximum likelihood estimates of the parameters α , k and λ .

AIC and log-likelihood

We calculate AIC value for each model with the same dataset, and the best model is the one with minimum AIC value. The value of AIC depends on the data Pines and Bombax, which leads to model selection uncertainty.

$$AIC=-2\log L(\hat{\theta}\mid x_i)+2k$$

where

$L(\hat{\theta}|x_i)$ = the maximized value of the likelihood function of the model, and where $\hat{\theta}$ are the parameter values that maximize the likelihood function;

x_i = the observed data;

k = the number of free parameters to be estimated.

Results and discussion

Summary of the data

The Bombax and Pine Height-Diameter data were extracted from the Forestry Research Institute of Nigeria's records, cleaned up and the summary statistics of the data was computed as presented in Tables 1 and 2.

Maximum likelihood approach

The above Table 3-5 shown the parameters estimation of EW, LBEW and ABEW distributions. We observed the comparison of the three distributions by their corresponding AIC and -2log-likelihood of each of the dataset pines and bombax. The ideal distribution is the one with the minimum AIC values (Figures 1-5).[7-15]

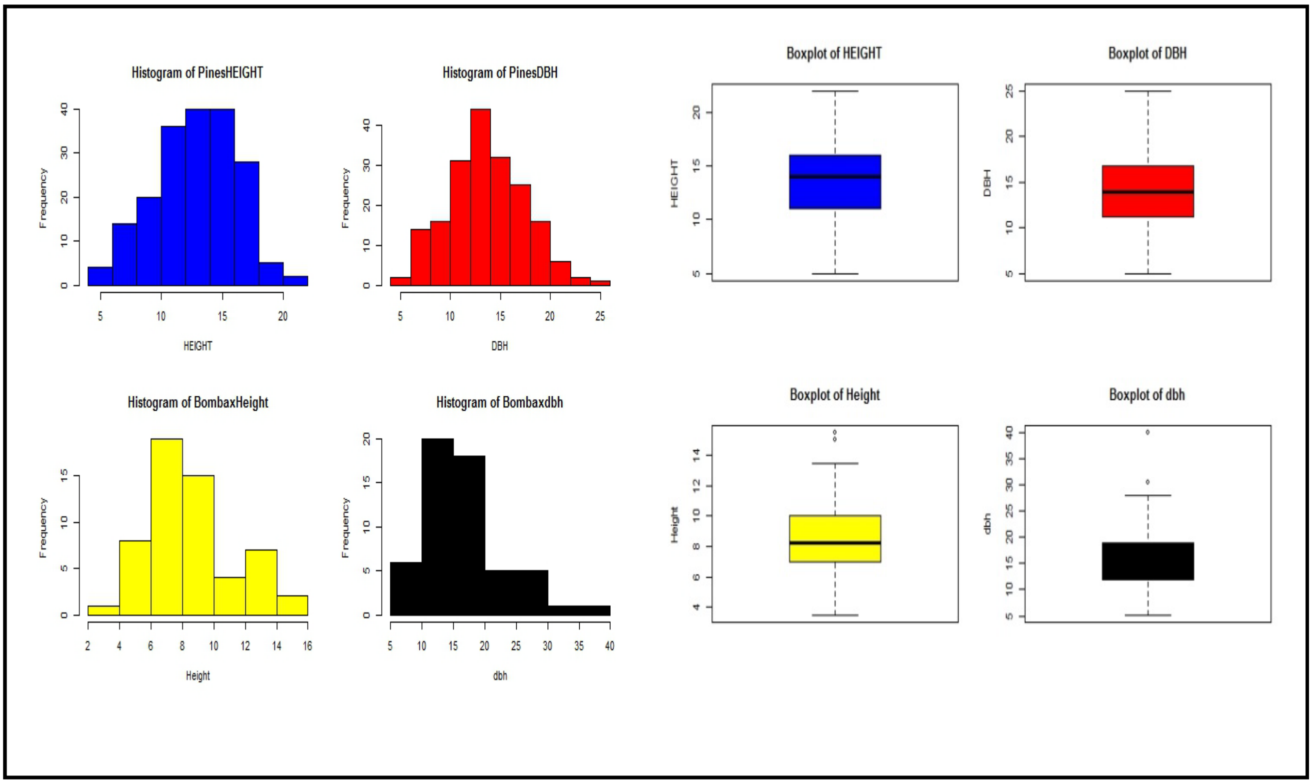

Figure 1 Histogram boxplot plot of bombax and pines H-D.

Citation: Oluwafemi OS, Olalekan DM. Length and area biased exponentiated weibull distribution based on forest inventories. *Biom Biostat Int J.* 2017;6(2):311–320. DOI: 10.15406/bbij.2017.06.00163

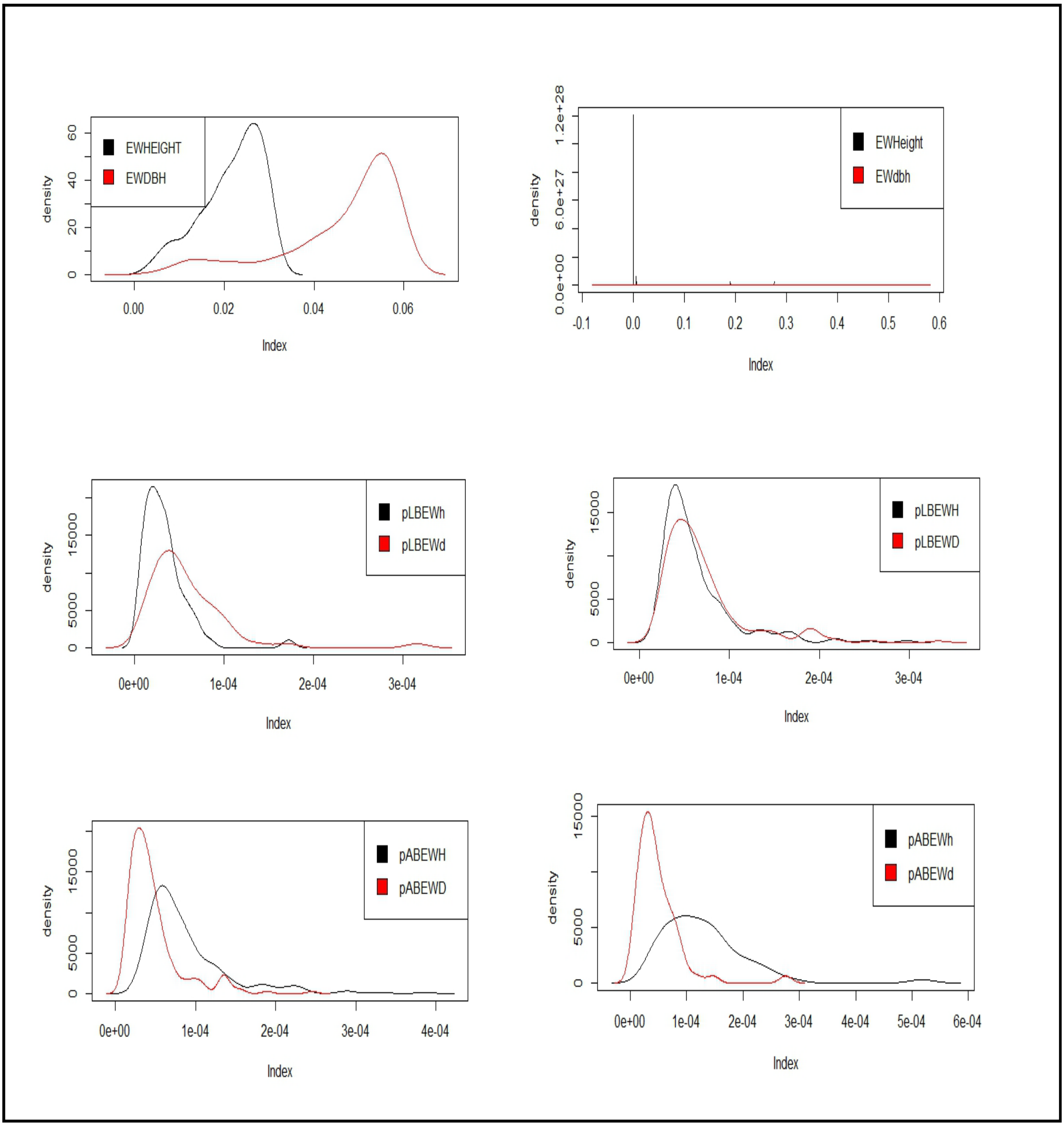

Figure 2 The probability distribution function of the EW, LBEW and ABEW distribution.

Table 1 Descriptive statistics of the data

	PinesHEIGHT	PinesDBH	Bombaxheight	Bombaxdbh
Mean	13.33333	13.87566	8.82142	16.47857
estimated stdev	3.38336	3.84751	2.67723	6.37982
estimated skewness	-0.14219	0.11252	0.6269	1.22949
estimated kurtosis	2.683	2.76197	2.85557	5.31719

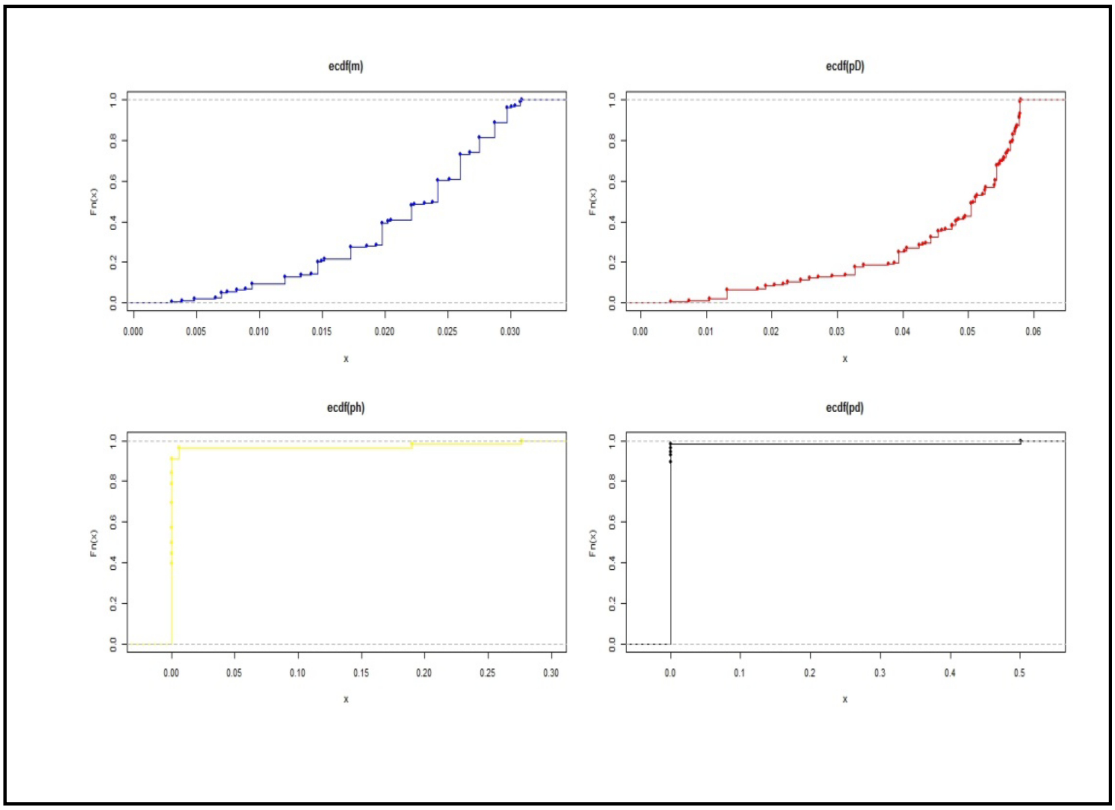

Figure 3 The cumulative distribution function of the EW distribution.

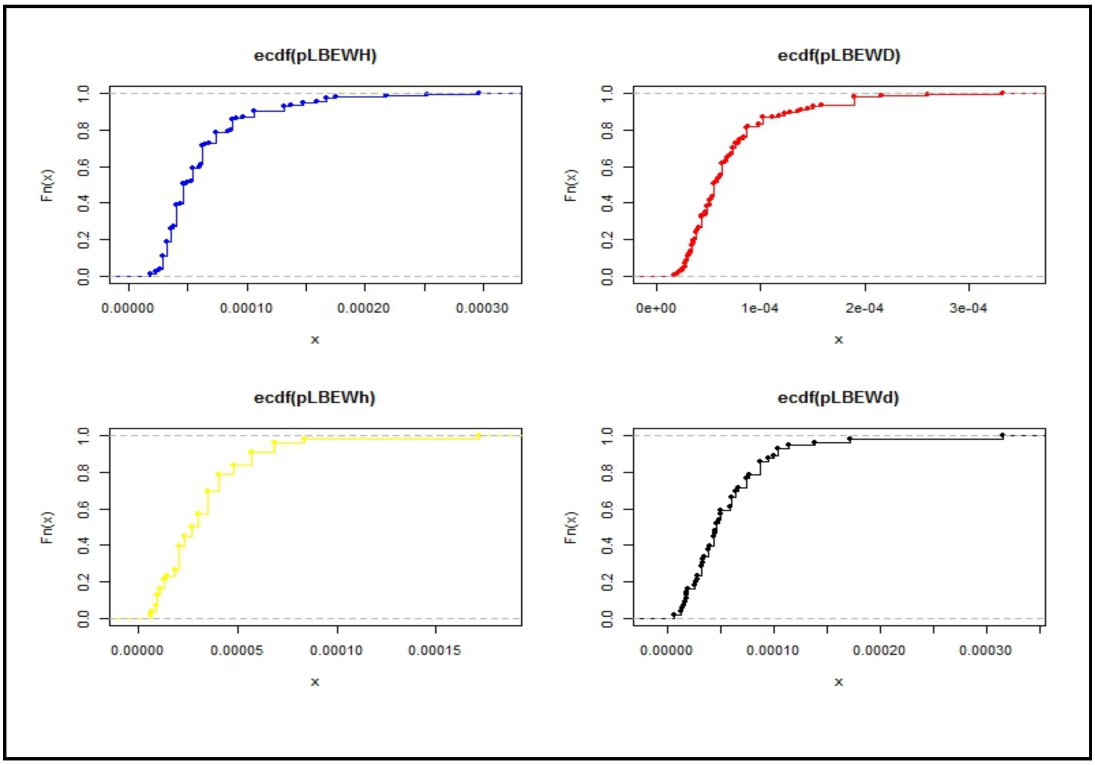

Figure 4 The cumulative distribution function of the LBEW distribution.

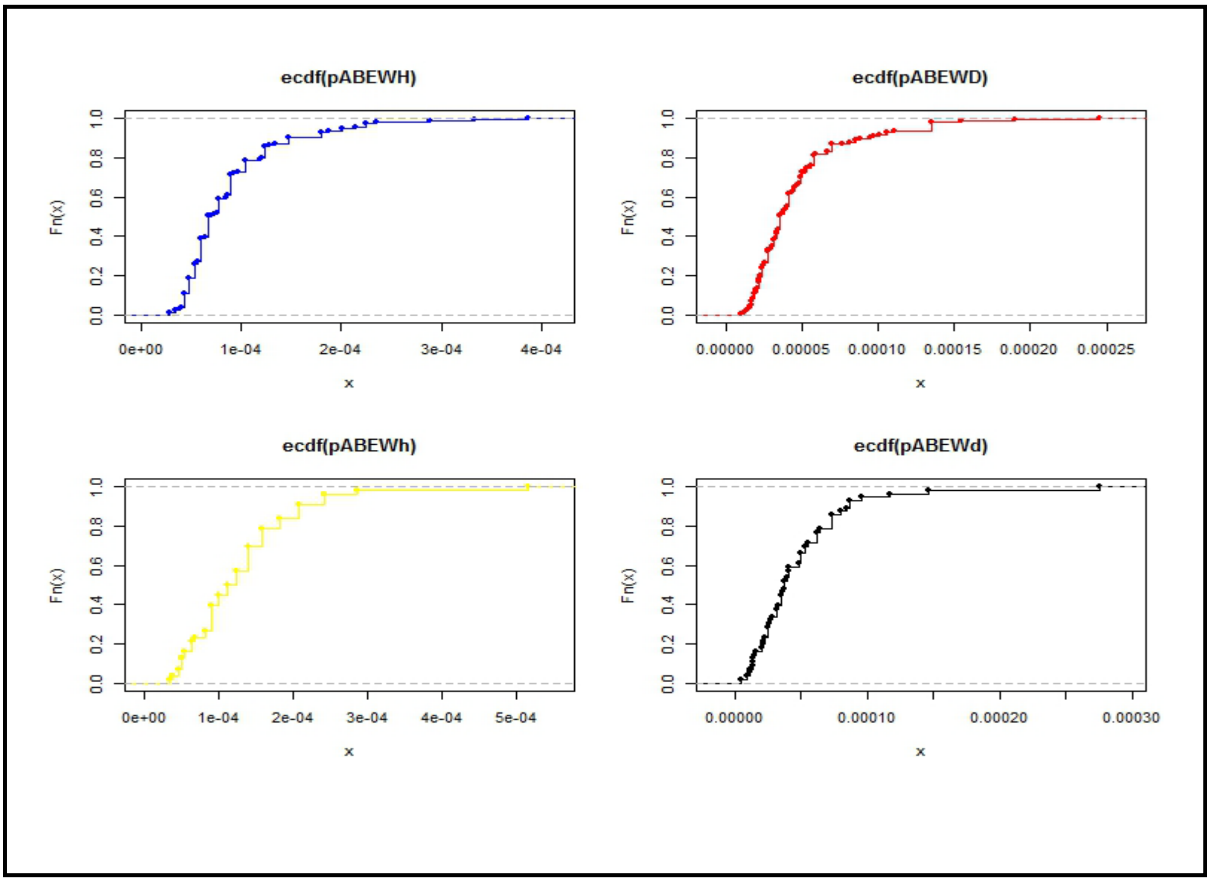

Figure 5 The cumulative distribution function of the ABEW distribution.

Table 2 Skewness and kurtosis of EW, LBEW and ABEW distribution

EW	**PinesHEIGHT**	**PinesDBH**	**Bombaxheight**	**Bombaxdbh**	
Mean	0.02165	0.046100	0.008641	0.008940	0.00685
estimated stdev	0.013384	0.044389	0.066902		
estimated skewness	-0.77246	-1.379207	5.268808	7.281359	
estimated kurtosis	2.757014	3.945957	29.65974	54.01818	
LBEW					
Mean	6.34E-05	7.07E-05	3.32E-05	5.88E-05	4.28E-05
estimated stdev	4.78E-05	2.62E-05	4.82E-05		
estimated skewness	2.408499	2.212744	2.871012	3.017609	
estimated kurtosis	10.28893	9.160734	15.27437	15.87016	
ABEW					
Mean	8.97E-05	4.76E-05	0.000130	4.88E-05	
estimated stdev	5.58E-05	3.51E-05	7.95E-05	4.20E-05	
estimated skewness	2.299099	2.336502	2.215247	3.142198	
estimated kurtosis	9.618972	9.925224	11.11609	16.72621	

Citation: Oluwafemi OS, Olalekan DM. Length and area biased exponentiated weibull distribution based on forest inventories. *Biom Biostat Int J.* 2017;6(2):311–320. DOI: 10.15406/bbij.2017.06.00163

Table 3 Parameters estimation of EW

EW	Pines		Bombax	
parameters	HEIGHT	DBH	Height	dbh
k	4.351	3.077	1.433	1.064
α	1.053	1.635	8.920	11.725
λ	14.468	13.542	4.352	5.747
AIC	1999.52	2098.4	538.40	719.45
-2loglik	993.76	1043.2	263.20	353.72

Table 4 Parameters estimation of LBEW

LBEW	Pines		Bombax	
parameters	HEIGHT	DBH	Height	dbh
k	0.05	0.05	0.05	0.05
α	0.264	0.261	0.263	0.261
λ	0.045	0.052	0.01	0.053
AIC	1629.07	1634.36	467.19	484.93
-2loglik	407.117	408.44	116.65	121.08

Table 5 Parameters estimation of ABEW

ABEW	Pines		Bombax	
parameters	HEIGHT	DBH	Height	dbh
k	0.050	0.050	0.050	0.050
α	0.262	0.262	0.267	0.263
λ	0.068	0.034	0.045	0.045
AIC	1533.11	1508.10	449.67	450.24
-2loglik	383.128	376.874	112.27	112.41

Conclusion

This study introduced a new distribution based on LBEW and ABEW. Some characteristics of the new distributions were obtained. Plots for the cumulative distribution function, pdf and tables with values of skewness and kurtosis were also provided. Height-Diameter (H-D) data on Bombax and Pines (Pinus caribeae) were used to demonstrate the application of the distributions. Estimation of parameters of EW, LBEW and ABEW distributions were done using the maximum likelihood approach and compared across the distributions using criteria like AIC and Log-likelihood. We therefore proposed that similar to Exponentiated Weibull distribution (EW), a better fitting of Bombax and Pines H-D data are possible by LBEW and ABEW distributions.

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Conflicts of interest

None.

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