

Ishita distribution and its applications

Abstract

In the present paper, a lifetime distribution named, “Ishita distribution” for modeling lifetime data from biomedical science and engineering has been proposed. Statistical properties of the distribution including its shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, stress-strength reliability have been discussed. The condition under which Ishita distribution is over-dispersed, equi-dispersed, and under-dispersed are presented along with the conditions under which Akash distribution, introduced by Shanker,¹ Lindley distribution, introduced by Lindley² and exponential distribution are over-dispersed, equi-dispersed and under-dispersed. Method of maximum likelihood estimation and method of moments have been discussed for estimating the parameter of the proposed distribution. Finally, the goodness of fit of the proposed distribution have been discussed and illustrated with two real lifetime data sets and the fit has been compared with exponential, Lindley and Akash distributions.

Keywords: akash distribution, lindley distribution, moments, dispersion, hazard rate function, mean residual life function, mean deviations, order statistics, stress-strength reliability, estimation of parameter, goodness of fit

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Introduction

The analyzing and modeling real lifetime data are crucial in many applied sciences including medicine, engineering, insurance and finance, amongst others. The two important one parameter lifetime distributions namely exponential and Lindley² are popular for modeling lifetime data from biomedical science and engineering. Recently, Shanker et al.,³ have conducted a comparative and critical study on the modeling of lifetime data from biomedical science and engineering using exponential and Lindley distributions and observed that there are many lifetime data where these two distributions are not suitable due to their shapes, nature of hazard rate functions, and mean residual life, amongst others. While searching a lifetime distribution which gives better fit than exponential and Lindley, Shanker¹ has introduced a lifetime distribution named Akash distribution and showed that Akash distribution gives much better fit than both exponential and Lindley distributions. Shanker et al.,⁴ have comparative study on the modeling of lifetime data using Akash, Lindley and exponential distribution and observed that there are several situations where these lifetime distributions are not suitable either from theoretical or applied point of view. Therefore, an attempt has been made in this paper to obtain a new lifetime distribution which is flexible than Akash, Lindley and exponential distributions for modeling lifetime data in reliability and in terms of its hazard rate shapes. The new one parameter lifetime distribution is based on a two- component mixture of an exponential distribution having scale parameter θ and a gamma distribution having shape parameter 3 and scale parameter θ with their mixing proportion $\frac{\theta^3}{\theta^3 + 2}$.

Lindley distribution, introduced by Lindley² has been defined by the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) as

$$f_1(x; \theta) = \frac{\theta^2}{\theta + 1} (1 + x) e^{-\theta x} ; x > 0, \theta > 0 \quad (1.1)$$

$$F_1(x; \theta) = 1 - \left[1 + \frac{\theta x}{\theta + 1} \right] e^{-\theta x} ; x > 0, \theta > 0 \quad (1.2)$$

It can be easily verified that the density (1.1) is a two-component mixture of an exponential (θ) distribution and a gamma ($2, \theta$) with their mixing proportion $\frac{\theta}{\theta + 1}$. Recent years much works have been done on Lindley distribution, its generalization and mixture with other distributions by several authors including Ghitany *et al.*,⁵ Zakerzadeh and Dolati,⁶ Mazucheli and Achcar,⁷ Bakouch *et al.*,⁸ Shanker and Mishra^{9,10} Shanker *et al.*,¹¹ Shanker and Amanuel,¹² Sankaran,¹³ are some among others.

Akash distribution, introduced by Shanker¹ has been defined by the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) as

$$f_2(x; \theta) = \frac{\theta^3}{\theta^2 + 2} (1 + x^2) e^{-\theta x} ; x > 0, \theta > 0 \quad (1.3)$$

$$F_2(x; \theta) = 1 - \left[1 + \frac{\theta x (\theta x + 2)}{\theta^2 + 2} \right] e^{-\theta x} ; x > 0, \theta > 0 \quad (1.4)$$

It can be easily verified that the Akash distribution is a two-component mixture of exponential (θ) distribution and a gamma ($3, \theta$) distribution with mixing proportion $\frac{\theta^2}{\theta^2 + 2}$. Shanker¹⁴ has obtained a Poisson mixture of Akash distribution named, “Poisson-Akash distribution (PAD) and discussed its properties, estimation of parameter and applications. Shanker et al.,¹⁵ have detailed study on modeling of count data from different fields of knowledge using Poisson-Akash distribution. Shanker and Shukla¹⁶ have obtained weighted Akash distribution and studied its statistical and mathematical properties, estimation of parameters and applications to model lifetime data. Shanker¹⁷ has also obtained a quasi Akash

distribution, studied its mathematical and statistical properties, estimation of parameters using both maximum likelihood estimation and method of moments and applications to model lifetime data.

The new one parameter lifetime distribution has been defined by its probability density function (p.d.f.)

$$f_3(x; \theta) = \frac{\theta^3}{\theta^3 + 2} (\theta + x^2) e^{-\theta x} ; x > 0, \theta > 0 \tag{1.5}$$

We would name this probability density function as, ‘‘Ishita distribution’’. It can be easily verified that the Ishita distribution is a two-component mixture of exponential (θ) distribution and a gamma ($3, \theta$) distribution with mixing proportion $\frac{\theta^3}{\theta^3 + 2}$.

The corresponding cumulative distribution function (c.d.f.) of (1.5) can be obtained as

$$F_3(x, \theta) = 1 - \left[1 + \frac{\theta x(\theta x + 2)}{\theta^3 + 2} \right] e^{-\theta x} ; x > 0, \theta > 0 \tag{1.6}$$

The graph of the p.d.f. and the c.d.f. of Ishita distribution for varying values of the parameter θ are shown in figures 1 and 2.

Statistical constants

The moment generating function of Ishita distribution (1.5) can be obtained as

$$\begin{aligned} M_X(t) &= \frac{\theta^3}{\theta^3 + 2} \int_0^\infty e^{-(\theta-t)x} (\theta + x^2) dx \\ &= \frac{\theta^3}{\theta^3 + 2} \left[\frac{\theta}{\theta - t} + \frac{2}{(\theta - t)^3} \right] \\ &= \frac{\theta^3}{\theta^3 + 2} \left[\sum_{k=0}^\infty \left(\frac{t}{\theta} \right)^k + \frac{2}{\theta^3} \sum_{k=0}^\infty \binom{k+2}{k} \left(\frac{t}{\theta} \right)^k \right] \\ &= \sum_{k=0}^\infty \frac{\theta^3 + (k+1)(k+2)}{\theta^3 + 2} \left(\frac{t}{\theta} \right)^k \end{aligned}$$

The r th moment about origin of Ishita distributon (1.5) is given by

$$\mu_r' = \frac{r! \left[\theta^3 + (r+1)(r+2) \right]}{\theta^r (\theta^3 + 2)} ; r = 1, 2, 3, \dots \tag{2.1}$$

The first four moments about origin are thus obtained as

$$\begin{aligned} \mu_1' &= \frac{\theta^3 + 6}{\theta(\theta^3 + 2)}, \mu_2' = \frac{2(\theta^3 + 12)}{\theta^2(\theta^3 + 2)} \\ \mu_3' &= \frac{6(\theta^3 + 20)}{\theta^3(\theta^3 + 2)}, \mu_4' = \frac{24(\theta^3 + 30)}{\theta^4(\theta^3 + 2)} \end{aligned}$$

Using relationship between moments about mean and the moments about origin, the moments about mean of Ishita distribution (1.5) can be obtained as

$$\begin{aligned} \mu_2 &= \frac{\theta^6 + 16\theta^3 + 12}{\theta^2(\theta^3 + 2)^2} \\ \mu_3 &= \frac{2(\theta^9 + 30\theta^6 + 36\theta^3 + 24)}{\theta^3(\theta^3 + 2)^3} \\ \mu_4 &= \frac{3(3\theta^{12} + 128\theta^9 + 408\theta^6 + 576\theta^3 + 240)}{\theta^4(\theta^3 + 2)^4} \end{aligned}$$

The coefficient of variation (CV), coefficient of skewness ($\sqrt{\beta_1}$), coefficient of kurtosis (β_2) and index of dispersion (γ) of Ishita distribution (1.5) are thus obtained as

$$\begin{aligned} CV &= \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^6 + 16\theta^3 + 12}}{\theta^3 + 6} \\ \sqrt{\beta_1} &= \frac{\mu_3}{\mu_2^{3/2}} = \frac{2(\theta^9 + 30\theta^6 + 36\theta^3 + 24)}{(\theta^6 + 16\theta^3 + 12)^{3/2}} \\ \beta_2 &= \frac{\mu_4}{\mu_2^2} = \frac{3(3\theta^{12} + 128\theta^9 + 408\theta^6 + 576\theta^3 + 240)}{(\theta^6 + 16\theta^3 + 12)^2} \\ \gamma &= \frac{\sigma^2}{\mu_1'} = \frac{\theta^6 + 16\theta^3 + 12}{\theta(\theta^3 + 2)(\theta^3 + 6)} \end{aligned}$$

The over-dispersion, equi-dispersion, and under-dispersion of Ishita distribution has been presented in table 1 along with Akash, Lindley and exponential distributions.

Table 1 Over-dispersion, equi-dispersion and under-dispersion of Ishita, Akash, Lindley and exponential distributions for the parameter θ

Lifetime Distributions	Over-Dispersion ($\mu < \sigma^2$)	Equi-Dispersion ($\mu = \sigma^2$)	Under-Dispersion ($\mu > \sigma^2$)
Ishita	$\theta < 1.535653152$	$\theta = 1.535653152$	$\theta > 1.535653152$
Akash	$\theta < 1.515400063$	$\theta = 1.515400063$	$\theta > 1.515400063$
Lindley	$\theta < 1.170086487$	$\theta = 1.170086487$	$\theta > 1.170086487$
Exponential	$\theta < 1$	$\theta = 1$	$\theta > 1$

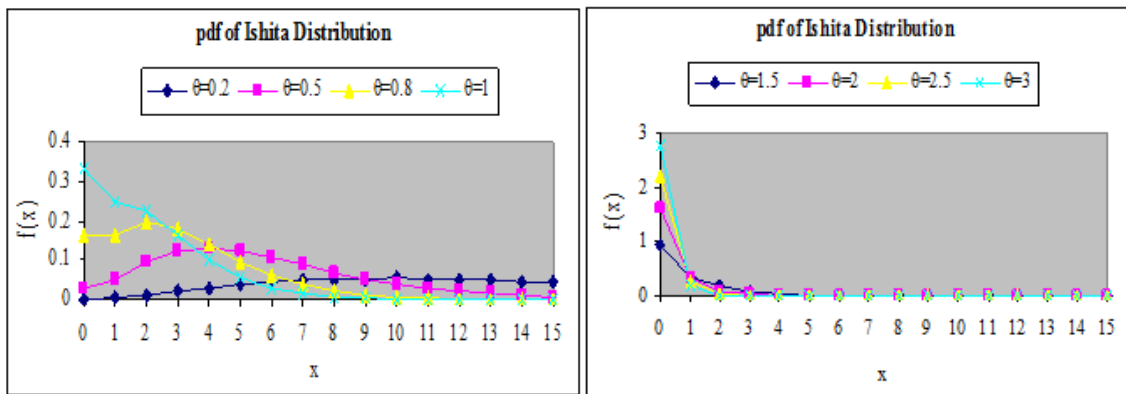


Figure 1 Graph of the pdf of Ishita distribution for varying values of the parameter θ .

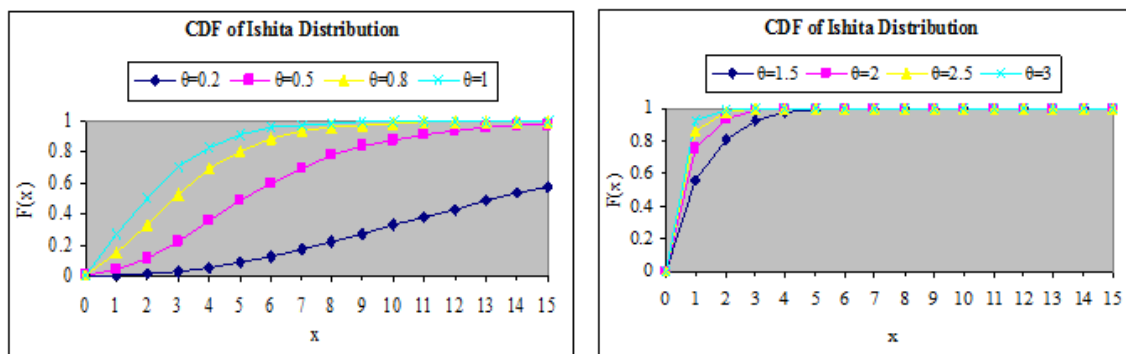


Figure 2 Graph of the cdf of Ishita distribution for varying values of the parameter θ .

Hazard rate function and mean residual life function

Let X be a continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$. The hazard rate function (also known as the failure rate function) $h(x)$ and the mean residual life function $m(x)$ of X are respectively defined as

$$h(x) = \lim_{\Delta x \rightarrow 0} \frac{P(X < x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)} \tag{3.1}$$

and

$$m(x) = E[X - x | X > x] = \frac{1}{1 - F(x)} \int_x^{\infty} [1 - F(t)] dt \tag{3.2}$$

The corresponding hazard rate function, $h(x)$ and the mean residual life function, $m(x)$ of the Ishita distribution (1.5) are thus obtained as

$$h(x) = \frac{\theta^3(\theta + x^2)}{\theta x(\theta x + 2) + (\theta^3 + 2)} \tag{3.3}$$

and

$$m(x) = \frac{\theta^3 + 2}{\left[\theta x(\theta x + 2) + (\theta^3 + 2) \right] e^{-\theta x}} \int_x^{\infty} \frac{\theta t(\theta t + 2) + (\theta^3 + 2)}{\theta^3 + 2} e^{-\theta t} dt$$

$$= \frac{\theta^2 x^2 + 4\theta x + (\theta^3 + 6)}{\theta \left[\theta x(\theta x + 2) + (\theta^3 + 2) \right]} \tag{3.4}$$

It can be easily verified that $h(0) = \frac{\theta^3}{\theta^3 + 2} = f(0)$ and $m(0) = \frac{\theta^3 + 6}{\theta(\theta^3 + 2)} = \mu_1'$. It is also obvious from the graphs of $h(x)$ and $m(x)$ that $h(x)$ is an increasing function for x and $\theta \leq 1$ and for $x \geq 1$ and $\theta > 1$ and decreasing function for $0 < x < 1$ and $\theta > 1$, whereas $m(x)$ is a decreasing function of x , and θ . The graph of the hazard rate function and mean residual life function of Ishita distribution (1.5) are shown in figures 3&4.

Stochastic ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

- i. Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x
- ii. Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x

- iii. Mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \leq m_Y(x)$ for all x
- iv. Likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x .

The following results due to Shaked and Shanthikumar¹⁸ are well known for establishing stochastic ordering of distributions

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \tag{4.1}$$

$$\Downarrow$$

$$X \leq_{st} Y$$

The Ishita distribution is ordered with respect to the strongest ‘likelihood ratio’ ordering as shown in the following theorem:

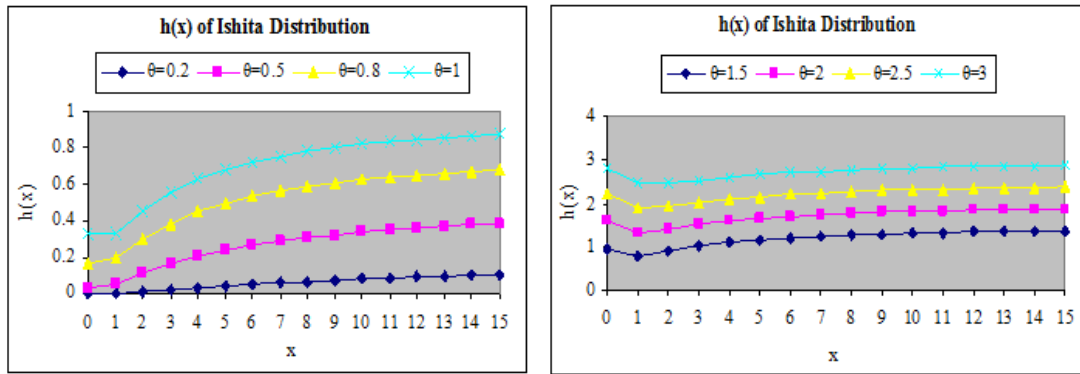


Figure 3 Graph of hazard rate function of Ishita distribution for varying values of parameter θ .

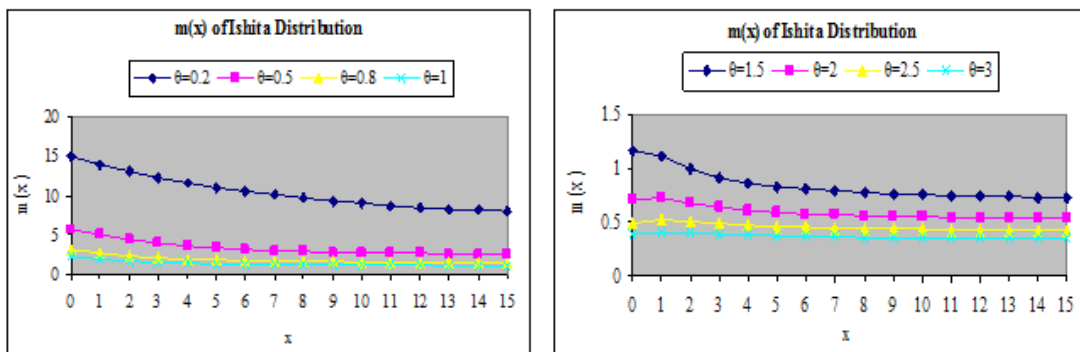


Figure 4 Graph of mean residual life function of Ishita distribution for varying values of parameter θ .

Theorem

Let $X \sim$ Ishita distribution (θ_1) and $Y \sim$ Ishita distribution (θ_2). If $\theta_1 > \theta_2$, then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof

We have

$$\frac{f_X(x, \theta)}{f_Y(x, \theta)} = \frac{\theta_1^3(\theta_2^3 + 2)}{\theta_2^3(\theta_1^3 + 2)} \left(\frac{\theta_1 + x^2}{\theta_2 + x^2} \right)^{-(\theta_1 - \theta_2)x} \quad ; x > 0$$

Now

$$\ln \frac{f_X(x, \theta)}{f_Y(x, \theta)} = \ln \left[\frac{\theta_1^3(\theta_2^3 + 2)}{\theta_2^3(\theta_1^3 + 2)} \right] + \ln \left(\frac{\theta_1 + x^2}{\theta_2 + x^2} \right)^{-(\theta_1 - \theta_2)x}$$

This gives

$$\frac{d}{dx} \ln \frac{f_X(x, \theta)}{f_Y(x, \theta)} = \frac{-2(\theta_1 - \theta_2)}{(\theta_1 + x^2)(\theta_2 + x^2)} - (\theta_1 - \theta_2)$$

Thus for $\theta_1 > \theta_2$, $\frac{d}{dx} \ln \frac{f_X(x, \theta)}{f_Y(x, \theta)} < 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Mean deviations

Generally the amount of scatter in a population is measured to some extent by the totality of deviations usually from their mean and median. These are known as the mean deviation about the mean and the mean deviation about the median defined as

$\delta_1(X) = \int_0^\infty |x - \mu| f(x) dx$ and $\delta_2(X) = \int_0^\infty |x - M| f(x) dx$, respectively, where $\mu = E(X)$ and $M = \text{Median}(X)$. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the following relationships

$$\delta_1(X) = \int_0^\mu (\mu - x) f(x) dx + \int_\mu^\infty (x - \mu) f(x) dx$$

$$\begin{aligned}
 &= \mu F(\mu) - \int_0^{\mu} x f(x) dx - \mu [1 - F(\mu)] + \int_{\mu}^{\infty} x f(x) dx \\
 &= 2\mu F(\mu) - 2\mu + 2 \int_{\mu}^{\infty} x f(x) dx \\
 &= 2\mu F(\mu) - 2 \int_0^{\mu} x f(x) dx
 \end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
 \delta_2(X) &= \int_0^M (M-x)f(x) dx + \int_M^{\infty} (x-M)f(x) dx \\
 &= M F(M) - \int_0^M x f(x) dx - M [1 - F(M)] + \int_M^{\infty} x f(x) dx \\
 &= -\mu + 2 \int_M^{\infty} x f(x) dx \\
 &= \mu - 2 \int_0^M x f(x) dx
 \end{aligned} \tag{5.2}$$

Using p.d.f. (1.5) and expression for the mean of Ishita distribution, we get

$$\int_0^{\mu} x f_3(x, \theta) dx = \mu \frac{\left\{ \theta^4 \mu + \theta^2 (\mu^3 + 1) + 3\theta^2 \mu^2 + 6(\theta \mu + 1) \right\} e^{-\theta \mu}}{\theta (\theta^3 + 2)} \tag{5.3}$$

$$\int_0^M x f_3(x, \theta) dx = \mu \frac{\left\{ \theta^4 M + \theta^3 (M^3 + 1) + 3\theta^2 M^2 + 6(\theta M + 1) \right\} e^{-\theta M}}{\theta (\theta^3 + 2)} \tag{5.4}$$

Using expressions from (5.1), (5.2), (5.3), and (5.4), the mean deviation about mean, $\delta_1(X)$ and the mean deviation about median, $\delta_2(X)$ of Ishita distribution are obtained as

$$\delta_1(X) = \frac{2 \left\{ \theta^2 \mu^2 + 4\theta \mu + (\theta^3 + 6) \right\} e^{-\theta \mu}}{\theta (\theta^3 + 2)} \tag{5.5}$$

$$\delta_2(X) = \frac{2 \left\{ \theta^4 M + \theta^3 (M^3 + 1) + 3\theta^2 M^2 + 6(\theta M + 1) \right\} e^{-\theta M}}{\theta (\theta^3 + 2)} - \mu \tag{5.6}$$

Order statistics

Let X_1, X_2, \dots, X_n be a random sample of size n from Ishita distribution (1.5). Let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ denote the corresponding order statistics. The p.d.f. and the c.d.f. of the k th order statistic, say $Y = X_{(k)}$ are given by

$$\begin{aligned}
 f_Y(y) &= \frac{n!}{(k-1)!(n-k)!} [F(y)]^{k-1} [1-F(y)]^{n-k} f(y) \\
 &= \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l [F(y)]^{k+l-1} f(y)
 \end{aligned}$$

and

$$\begin{aligned}
 F_Y(y) &= \sum_{j=k}^n \binom{n}{j} [F(y)]^j [1-F(y)]^{n-j} \\
 &= \sum_{j=k}^n \sum_{l=0}^{n-j} \binom{n-j}{l} (-1)^l [F(y)]^{j+l},
 \end{aligned}$$

respectively, for $k=1, 2, 3, \dots, n$.

Thus, the p.d.f. and the c.d.f. of k th order statistic of Ishita distribution (1.5) are given by

$$f_Y(y) = \frac{n! \theta^3 (\theta + x^2) e^{-\theta x}}{(\theta^3 + 2)(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} \times \left[1 - \frac{\theta x (\theta x + 2) + (\theta^3 + 2)}{\theta^3 + 2} e^{-\theta x} \right]^{k+l-1}$$

and

$$F_Y(y) = \sum_{j=k}^n \sum_{l=0}^{n-j} \binom{n-j}{l} (-1)^l \left[1 - \frac{\theta x (\theta x + 2) + (\theta^3 + 2)}{\theta^3 + 2} e^{-\theta x} \right]^{j+l}$$

Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves Bonferroni¹⁹ and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx = \frac{1}{p\mu} \left[\int_0^{\infty} x f(x) dx - \int_q^{\infty} x f(x) dx \right] = \frac{1}{p\mu} \left[\mu - \int_q^{\infty} x f(x) dx \right] \tag{7.1}$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x f(x) dx = \frac{1}{\mu} \left[\int_0^{\infty} x f(x) dx - \int_q^{\infty} x f(x) dx \right] = \frac{1}{\mu} \left[\mu - \int_q^{\infty} x f(x) dx \right] \tag{7.2}$$

respectively or equivalently

$$B(p) = \frac{1}{p\mu} \int_0^p F^{-1}(x) dx \tag{7.3}$$

and

$$L(p) = \frac{1}{\mu} \int_0^p F^{-1}(x) dx \tag{7.4}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$.

The Bonferroni and Gini indices are thus defined as

$$B = 1 - \int_0^1 B(p) dp \tag{7.5}$$

And

$$G = 1 - 2 \int_0^1 L(p) dp \tag{7.6}$$

respectively. Using p.d.f. (1.5), we get

$$\int_q^\infty x f_3(x, \theta) dx = \frac{\left\{ \theta^4 q + \theta^3 (q^3 + 1) + 3\theta^2 q^2 + 6(\theta q + 1) \right\} e^{-\theta q}}{\theta (\theta^3 + 2)} \quad (7.7)$$

Now using equation (7.7) in (7.1) and (7.2), we get

$$B(p) = \frac{1}{p} \left[1 - \frac{\left\{ \theta^4 q + \theta^3 (q^3 + 1) + 3\theta^2 q^2 + 6(\theta q + 1) \right\} e^{-\theta q}}{\theta^3 + 6} \right] \quad (7.8)$$

and

$$L(p) = 1 - \frac{\left\{ \theta^4 q + \theta^3 (q^3 + 1) + 3\theta^2 q^2 + 6(\theta q + 1) \right\} e^{-\theta q}}{\theta^3 + 6} \quad (7.9)$$

Now using equations (7.8) and (7.9) in (7.5) and (7.6), the Bonferroni and Gini indices of Ishita distribution (1.5) are obtained as

$$B = 1 - \frac{\left\{ \theta^4 q + \theta^3 (q^3 + 1) + 3\theta^2 q^2 + 6(\theta q + 1) \right\} e^{-\theta q}}{\theta^3 + 6} \quad (7.10)$$

$$G = \frac{2 \left\{ \theta^4 q + \theta^3 (q^3 + 1) + 3\theta^2 q^2 + 6(\theta q + 1) \right\} e^{-\theta q}}{\theta^2 + 6} - 1 \quad (7.11)$$

Renyi entropy

An entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy [20]. If X is a continuous random variable having probability density function $f(\cdot)$, then Renyi entropy is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma} \log \left\{ \int f^\gamma(x) dx \right\}$$

where $\gamma > 0$ and $\gamma \neq 1$.

Thus, the Renyi entropy for the Ishita distribution (1.5) can be obtained as

$$\begin{aligned} T_R(\gamma) &= \frac{1}{1-\gamma} \log \left[\int_0^\infty \frac{\theta^{3\gamma}}{(\theta^3 + 2)^\gamma} (\theta + x^2)^\gamma e^{-\theta\gamma x} dx \right] \\ &= \frac{1}{1-\gamma} \log \left[\int_0^\infty \frac{\theta^{3\gamma}}{(\theta^3 + 2)^\gamma} \theta^\gamma \left(1 + \frac{x^2}{\theta} \right)^\gamma e^{-\theta\gamma x} dx \right] \\ &= \frac{1}{1-\gamma} \log \left[\int_0^\infty \frac{\theta^{4\gamma}}{(\theta^3 + 2)^\gamma} \sum_{j=0}^\infty \binom{\gamma}{j} \left(\frac{x^2}{\theta} \right)^j e^{-\theta\gamma x} dx \right] \\ &= \frac{1}{1-\gamma} \log \left[\int_0^\infty \frac{\theta^{4\gamma-j}}{(\theta^3 + 2)^\gamma} \sum_{j=0}^\infty \binom{\gamma}{j} x^{2j} e^{-\theta\gamma x} dx \right] \\ &= \frac{1}{1-\gamma} \log \left[\sum_{j=0}^\infty \binom{\gamma}{j} \frac{\theta^{4\gamma-j}}{(\theta^3 + 2)^\gamma} \int_0^\infty e^{-\theta\gamma x} x^{2j+1} dx \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{1-\gamma} \log \left[\sum_{j=0}^\infty \binom{\gamma}{j} \frac{\theta^{4\gamma-j} \Gamma(2j+1)}{(\theta^3 + 2)^\gamma (\theta\gamma)^{2j+1}} \right] \\ &= \frac{1}{1-\gamma} \log \left[\sum_{j=0}^\infty \binom{\gamma}{j} \frac{\theta^{4\gamma-3j-1} \Gamma(2j+1)}{(\theta^3 + 2)^\gamma (\gamma)^{2j+1}} \right] \end{aligned}$$

Stress-strength reliability

The stress-strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y . When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily till $X > Y$. Therefore, $R = P(Y < X)$ is a measure of component reliability and in statistical literature it is known as stress-strength parameter. It has wide applications in almost all areas of knowledge especially in engineering such as structures, deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels etc.

Let X and Y be independent strength and stress random variables having Ishita distribution (1.5) with parameter θ_1 and θ_2 respectively. Then the stress-strength reliability R of Ishita distribution can be obtained as

$$\begin{aligned} R &= P(Y < X) = \int_0^\infty P(Y < X | X = x) f_X(x) dx \\ &= \int_0^\infty f_3(x, \theta_1) F_3(x, \theta_2) dx \\ &= 1 - \frac{\theta_1^3 \left[24\theta_2^2 + 12\theta_2(\theta_1 + \theta_2) + 2(\theta_2^3 + \theta_1\theta_2^2 + 2)(\theta_1 + \theta_2)^2 \right. \\ &\quad \left. + 2\theta_1\theta_2(\theta_1 + \theta_2)^3 + \theta_1(\theta_2^3 + 2)(\theta_1 + \theta_2)^4 \right]}{(\theta_1^3 + 2)(\theta_2^3 + 2)(\theta_1 + \theta_2)^5} \end{aligned}$$

Estimation of parameter

Maximum likelihood estimate (MLE)

Let $(x_1, x_2, x_3, \dots, x_n)$ be a random sample from Ishita distribution (1.5). The likelihood function, L of (1.5) is given by

$$L = \left(\frac{\theta^3}{\theta^3 + 2} \right)^n \prod_{i=1}^n (\theta + x_i^2) e^{-n\theta \bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L = n \ln \left(\frac{\theta^3}{\theta^3 + 2} \right) + \sum_{i=1}^n \ln(\theta + x_i^2) - n\theta \bar{x}$$

Now

$$\frac{d \ln L}{d\theta} = \frac{6n}{\theta(\theta^3 + 2)} + \sum_{i=1}^n \frac{1}{\theta + x_i^2} - n\bar{x}$$

where \bar{x} is the sample mean.

The maximum likelihood estimate, $\hat{\theta}$ of θ is the solution of the equation $\frac{d \ln L}{d \theta} = 0$ and it can be obtained by solving the following non-linear equation

$$\frac{6n}{\theta(\theta^3 + 2)} + \sum_{i=1}^n \frac{1}{\theta + x_i^2} - n\bar{x} = 0$$

Method of moment estimate (MOME)

Equating the population mean of the Ishita distribution (1.5) to the corresponding sample mean, the method of moment estimate (MOME) $\hat{\theta}$ of θ is the solution of the following non-linear equation

$$\bar{x} \theta^4 - \theta^3 + 2\theta\bar{x} - 6 = 0, \text{ where } \bar{x} \text{ is the sample mean.}$$

Goodness of fit of Ishita distribution

The goodness of fit of Ishita distribution has been done on several lifetime data sets. In this section, we present the goodness of fit of Ishita distribution using maximum likelihood estimate of the parameter on two data sets and the fit has been compared with Akash, Lindley and exponential distributions. For testing the goodness of fit of Ishita distribution over exponential, Lindley and Akash distributions, following two data sets have been considered.

Data set 1: The second data set is the strength data of glass of the aircraft window reported by Fuller et al.,²¹

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381

Data Set 2: The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm, Bader and Priest²²

1.312	1.314	1.479	1.552	1.700	1.803	1.861
1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.027	2.055	2.063	2.098	2.140	2.179	2.224
2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.359	2.382	2.382	2.426	2.434	2.435	2.478
2.490	2.511	2.514	2.535	2.554	2.566	2.570
2.586	2.629	2.633	2.642	2.648	2.684	2.697
2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.880	2.954	3.012	3.067	3.084	3.090
3.096	3.128	3.233	3.433	3.585	3.585	

In order to compare Ishita, Akash, Lindley and exponential distributions, values of $-2 \ln L$, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion) and K-S Statistic (Kolmogorov-Smirnov Statistic) for two real data sets have been computed and presented in table 2. The formulae for computing AIC, AICC, BIC, and K-S Statistic are as follows:

$$AIC = -2 \ln L + 2k, \quad AICC = AIC + \frac{2k(k+1)}{(n-k-1)}, \quad BIC = -2 \ln L + k \ln n$$

$$K-S = \sup_x |F_n(x) - F_0(x)|, \text{ where } k = \text{the number of parameters,}$$

n = the sample size and $F_n(x)$ is the empirical distribution function.

Table 2 MLEs of $\hat{\theta}$, S.E. ($\hat{\theta}$), $-2 \ln L$, AIC, AICC, BIC, and K-S Statistic of the fitted distributions of data set 1 and 2

	Distributions	MLE of $\hat{\theta}$	S.E. ($\hat{\theta}$)	$-2 \ln L$	AIC	AICC	BIC	K-S
Data 1	Ishita	0.0973	0.0100	240.48	242.48	242.62	243.91	0.297
	Akash	0.0971	0.0101	240.68	242.68	242.82	244.11	0.298
	Lindley	0.0630	0.0080	253.98	255.98	256.12	257.41	0.365
	Exponential	0.0324	0.0058	274.52	276.52	276.66	277.95	0.458
Data 2	Ishita	0.9315	0.0560	223.14	225.14	225.20	227.37	0.331
	Akash	0.9647	0.0646	224.27	226.27	226.33	228.50	0.362
	Lindley	0.6545	0.0580	238.38	240.38	240.44	242.61	0.401
	Exponential	0.4079	0.0491	261.73	263.73	263.79	265.96	0.448

The best distribution corresponds to lower values of $-2 \ln L$, AIC, AICC, BIC, and K-S statistic.

It can be easily seen from above table that Ishita distribution gives better fit than exponential, Lindley and Akash distributions and hence Ishita distribution should be preferred to exponential, Lindley and Akash distributions for modeling lifetime data from biomedical science and engineering.

Concluding remarks

A lifetime distribution named, ‘‘Ishita distribution’’ for modeling lifetime data from biomedical science and engineering has been proposed and its various statistical and mathematical properties including its shape, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measure and stress-strength reliability have been studied. The conditions of over-

dispersed, equi-dispersed, and under-dispersed of Ishita distribution has been presented along with Akash, Lindley and exponential distributions. The estimation of parameter has been discussed using both maximum likelihood estimation and method of moments. The goodness of fit of Ishita distribution has been discussed and illustrated with two real lifetime data sets and it has been shown that it gives better fit than exponential, Lindley and Akash distributions.

NOTE: The paper is named Ishita distribution in the name of Ishita Shukla, a lovely daughter of second author Dr. Kamlesh Kumar Shukla, Department of Statistics, Eritrea Institute of Technology, Asmara, Eritrea.

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Conflict of interest

None.

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