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On modeling of lifetime data using three-parameter generalized lindley and generalized gamma distributions

Abstract

The analysis and modeling of lifetime data are crucial in almost all applied sciences including behavioral sciences, medicine, insurance, engineering, and finance, amongst others. In this paper an attempt has been made for comparative study of generalized Lindley distribution (GLD) introduced by Zakerzadeh & Dolati[1] and generalized gamma distribution (GGD) introduced by Stacy[2] for modeling lifetime data from different fields of knowledge. The goodness of fit for both GLD and GGD , based on maximum likelihood estimates, shows that GGD gives much closer fit than GLD in majority of data sets and hence GGD can be considered as an important tool for modeling lifetime data over GLD.

Keywords: generalized lindley distribution, generalized gamma distribution, lifetime data, estimation of parameter, goodness of fit

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Introduction

In reliability analysis the time to the occurrence of event of interest is the lifetime or survival time or failure time. The event may be failure of a piece of equipment, death of a person, development (or remission) of symptoms of disease, health code violation (or compliance). The statistical analysis and the modeling of lifetime data are crucial for statisticians and research workers in almost all applied sciences including behavioral sciences, engineering, medical science/biological science, insurance and finance, amongst others.

A number of lifetime distributions are available for modeling lifetime data in statistics including exponential distribution, gamma distribution, Lindley distribution, Weibull distribution and their generalizations, some amongst others.

In this paper various lifetime data have been considered for modeling using three- parameter generalized Lindley distribution (GLD) and generalized gamma distribution (GGD) because gamma, Lindley and exponential distributions are particular cases of GLD whereas gamma, Weibull and exponential distributions are particular cases of GGD.

Generalized lindley distribution

The probability density function of three-parameter generalized Lindley distribution (GLD) introduced by Zakerzadeh & Dolati[1] having parameters α, β, and θ is given by

$$f_1(x;\alpha,\beta,\theta)=\frac{\theta^{\alpha+1}}{(\beta+\theta)}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha+\beta x)e^{-\theta x};\, x>0,\alpha>0,\beta>0,\theta>0 \quad (2.1)$$

Clearly the gamma distribution, the Lindley[3] distribution and the exponential distribution are particular cases of (2.1) for $(\beta=0)$, $(\alpha=1,\beta=0)$ and $(\alpha=1,\beta=0)$ respectively. The discussion about its properties, estimation of parameters and applications are available in Zakerzadeh & Dolati.[1] Ghitany et al.,[4] have detailed study about various properties of Lindley distribution, estimation of parameter and application for modeling waiting time data in a bank. Shanker et al.,[5] have detailed and comparative study about modeling of lifetime data using one parameter Lindley and exponential distributions.

The corresponding distribution function of the GLD can be obtained as

$$F_1(x;\alpha,\beta,\theta)=1-\frac{\alpha(\beta+\theta)\Gamma(\alpha,\theta x)+\beta(\theta x)^{\alpha}e^{-\theta x}}{(\beta+\theta)\Gamma(\alpha+1)};$$

$$x>0,\alpha>0,\beta>0,\theta>0 \quad (2.2)$$

Where $\Gamma(\alpha,z)$ is the upper incomplete gamma function defined as

$$(\alpha,z)=\int_z^{\infty}e^{-y}y^{\alpha-1}dy;\alpha>0,\, z\geq 0 \quad (2.3)$$

Recently Shanker[6] has detailed study about GLD and obtained expressions for coefficient of variation, skewness, kurtosis and index of dispersion. Shanker[6] has also studied its hazard rate function and the mean residual life function.

Generalized gamma distribution

The probability density function of three-parameter generalized gamma distribution (GGD) introduced by Stacy[2] having parameters α, β, and θ is given by

$$f_2(x;\alpha,\beta,\theta)=\frac{\beta\theta^{\alpha}}{\Gamma(\alpha)}x^{\beta\alpha-1}e^{-\theta x^{\beta}};\, x>0,\alpha>0,\beta>0,\theta>0 \quad (3.1)$$

Where α and β are the shape parameter and θ is the scale parameter. Clearly the gamma distribution, the Weibull distribution and the exponential distribution are particular cases of (3.1) for $(\beta=1)$, $(\alpha=1)$ and $(\alpha=\beta=1)$ respectively. Detailed discussion about GGD is available in Stacy[2] and parametric estimation for the GGD is available in Stacy & Mihram.[7]

The cumulative distribution function of the GGD is thus given by

$$F_2(x;\alpha,\beta,\theta)=1-\frac{\Gamma(\alpha,\theta x^{\beta})}{\Gamma(\alpha)} \quad (3.2)$$

Where $\Gamma(\alpha,z)$ is the upper incomplete gamma function defined in (2.3)

Maximum likelihood estimation

Maximum likelihood estimates of the parameters of GLD

Assuming $(x_1,x_2,x_3,\dots,x_n)$ be a random sample of size n from GLD (2.1), the likelihood function, L of GLD is given by

$$L=\left(\frac{\theta^{\alpha+1}}{\beta+\theta}\right)^n\frac{1}{\left(\Gamma(\alpha+1)\right)^n}\prod_{i=1}^{n}x_i^{\alpha-1}(\alpha+\beta x_i)e^{-n\theta\bar{x}}$$; $\bar{x}$ being the sample mean

The natural log likelihood function is thus obtained as

The maximum likelihood estimate (MLE) $\hat{\theta},\hat{\alpha},\hat{\beta}$ of parameters θ,α,β of GLD can be obtained by solving the natural log likelihood equation using R software (Package Stat 4).

Maximum likelihood estimates of the parameters of GGD

Let $(x_1,x_2,x_3,\dots,x_n)$ be a random sample of size n from GGD (3.1). Then

$$f(x_i)=\frac{\beta\theta^{\alpha}}{\Gamma(\alpha)}(x_i)^{\beta\alpha-1}e^{-\theta(x_i)^{\beta}}$$

This gives

$$\ln f(x_i)=\ln\beta+\alpha\ln\theta-\ln\left(\Gamma(\alpha)\right)+(\beta\alpha-1)\ln(x_i)-\theta(x_i)^{\beta}$$

Thus the natural log likelihood function of the GGD is given by

$$\ln L=n\left[\ln\beta+\alpha\ln\theta-\ln\left(\Gamma(\alpha)\right)\right]+(\beta\alpha-1)\sum_{i=1}^{n}\ln(x_i)-\theta\sum_{i=1}^{n}x_i^{\beta}$$

The maximum likelihood estimate (MLE) $\hat{\theta},\hat{\alpha},\hat{\beta}$ of parameters θ,α,β of GGD can be obtained by solving the natural log likelihood equation using R software (Package Stat 4).

Goodness of fit and applications

In this section, the goodness of fit and applications of GLD and GGD have been discussed for several lifetime data In order to compare GLD and GGD, $-2\ln L$ and K-S Statistics Kolmogorov-Smirnov Statistics) for eighteen data sets have been computed and presented in Table 1.

The formula for K-S Statistics is defined as follow:

$K\text{-}S=\sup_x\left|F_n(x)-F_0(x)\right|$, where $F_n(x)$ is the empirical distribution function. The best distribution corresponds to lower values of $-2\ln L$ and K-S statistics and higher p-values. It is clear from the goodness of fit of GLD and GGD that in most of the data sets except Data sets (1-18) GGD gives much closer fit than GLD for modeling lifetime data.

Conclusion

The modeling and analysis of lifetime data using lifetime distributions are crucial in almost all applied sciences including behavioral sciences, medicine, insurance, engineering, and finance, amongst others. In this paper an attempt has been made to have a comparative study on modeling of lifetime data on eighteen data sets using three parameters generalized Lindley distribution (GLD) introduced by Zakerzadeh & Dolati[1] and generalized gamma distribution (GGD) introduced by Stacy.[2] Maximum likelihood estimates have been used for fitting both GLD and GGD. The goodness of fit for both GLD and GGD shows that GGD gives much closer fit than GLD in majority of data sets and hence GGD can be considered as an important tool for modeling lifetime data over GLD.

Data Set 1 The data set represents the strength of 1.5cm glass fibers measured at the National Physical Laboratory, England. Unfortunately, the units of measurements are not given in the paper, and they are taken from Smith & Naylor[8]

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81
2.00	0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76
1.82	2.01	0.77	1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69
1.76	1.84	2.24	0.81	1.13	1.29	1.48	1.51	1.55	1.61	1.62	1.66
1.70	1.77	1.84	0.84	1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67
1.70	1.78	1.89									

Data Set 2 The data is given by Birnbaum & Saunders[9] on the fatigue life of 6061 –T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data ($\times 10^{-3}$) are presented below (after subtracting 65)

5	25	31	32	34	35	38	39	39	40	42	43
43	43	44	44	47	47	48	49	49	49	51	54
55	55	55	56	56	56	58	59	59	59	59	59
63	63	64	64	65	65	65	66	66	66	66	66
67	67	67	68	69	69	69	69	71	71	72	73
73	73	74	74	76	76	77	77	77	77	77	77
79	79	80	81	83	84	84	86	86	87	90	91
92	92	92	92	93	94	97	98	98	99	101	103
105	109	136	147								

Data Set 3 The data set is from Lawless.[10] The data given arose in tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life tests and they are

17.88	28.92	33.00	41.52	42.12	45.60	48.80	51.84	51.96	54.12	55.56	67.80
68.44	68.64	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Data Set 4 The data is from Picciotto[11] and arose in test on the cycle at which the Yarn failed. The data are the number of cycles until failure of the yarn and they are

86	146	251	653	98	249	400	292	131	169	175	176
76	264	15	364	195	262	88	264	157	220	42	321
180	198	38	20	61	121	282	224	149	180	325	250
196	90	229	166	38	337	65	151	341	40	40	135
597	246	211	180	93	315	353	571	124	279	81	186
497	182	423	185	229	400	338	290	398	71	246	185
188	568	55	55	61	244	20	284	393	396	203	829
239	236	286	194	277	143	198	264	105	203	124	137
135	350	193	188								

Data Set 5 This data represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal[12]

10	33	44	56	59	72	74	77	92	93	96	100
100	102	105	107	107	108	108	108	109	112	113	115
116	120	121	122	122	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176	183	195	196
197	202	213	215	216	222	230	231	240	245	251	253
254	254	278	293	327	342	347	361	402	432	458	555

Data Set 6 This data is related with behavioral sciences, collected by Balakrishnan N et al. [13]: The scale "General Rating of Affective Symptoms for Preschoolers (GRASP)" measures behavioral and emotional problems of children, which can be classified with depressive condition or not according to this scale. A study conducted by the authors in a city located at the south part of Chile has allowed collecting real data corresponding to the scores of the GRASP scale of children with frequency in parenthesis, which are

19(16)	20(15)	21(14)	22(9)	23(12)	24(10)	25(6)	
26(9)	27(8)	28(5)	29(6)	30(4)	31(3)	32(4)	
33	34	35(4)	36(2)	37(2)	39	42	44

Data Set 7 The data set reported by Efron[14] represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiotherapy (RT).

6.53	7	10.42	14.48	16.10	22.70	34	41.55	42	45.28	49.40	53.62
63	64	83	84	91	108	112	129	133	133	139	140
140	146	149	154	157	160	160	165	146	149	154	157
160	160	165	173	176	218	225	241	248	273	277	297
405	417	420	440	523	583	594	1101	1146	1417		

Data Set 8 The data set reported by Efron[14] represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT)

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47	68.46
78.26	74.47	81.43	84	92	94	110	112	119	127	130	133
140	146	155	159	173	179	194	195	209	249	281	319
339	432	469	519	633	725	817	1776				

Citation: Shanker R, Shukla KK. On modeling of lifetime data using three-parameter generalized lindley and generalized gamma distributions. *Biom Biostat Int J.* 2016;4(7):283–288. DOI: 10.15406/bbij.2016.04.00117

Data set 9 This data set represents remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee & Wang[15]

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98
6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28
9.74	14.76	6.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25
8.37	12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69				

Data Set 10 This data set is given by Linhart & Zucchini [16] which represents the failure times of the air conditioning system of an airplane

23	261	87	7	120	14	62	47	225	71	246	21
42	20	5	12	120	11	3	14	71	11	14	11
16	90 1	16	52	95							

Data Set 11 This data set used by Bhaumik et al.,[17] is vinyl chloride data obtained from clean up gradient monitoring wells in mg/l

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8	0.8	0.4	0.6
0.9	0.4	2	0.5	5.3	3.2	2.7	2.9	2.5	2.3	1	0.2
0.1	0.1	1.8	0.9	2	4	6.8	1.2	0.4	0.2		

Data set 12 This data set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by Ghitany et al.,[4] for fitting the Lindley[3] distribution

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1
3.2	3.3	3.5	3.6	4	4.1	4.2	4.2	4.3	4.3	4.4	4.4
4.6	4.7	4.7	4.8	4.9	4.9	5	5.3	5.5	5.7	5.7	6.1
6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6
7.7	8	8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11	11	11.1	11.2	11.2	11.5	11.9	12.4
12.5	12.9	13	13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3
17.3	18.1	18.2	18.4	18.9	19	19.9	20.6	21.3	21.4	21.9	23
27	31.6	33.1	38.5								

Data Set 13 This data is for the times between successive failures of air conditioning equipment in a Boeing 720 airplane, Proschan[18]

74	57	48	29	502	12	70	21	29	386	59	27
153	26	326									

Data set 14 This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross & Clark[19]

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7	4.1	1.8
1.5	1.2	1.4	3	1.7	2.3	1.6	2				

Data Set 15 This data set is the strength data of glass of the aircraft window reported by Fuller et al.,[20]

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77
26.78	27.05	27.67	29.9	31.11	33.2	33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29	45.381					

Data Set 16 The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm, Bader & Priest[21]

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997
2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585			

Data Set 17 The following data set represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test, Lawless.[10]

1.4	5.1	6.3	10.8	12.1	18.5	19.7	22.2	23.0	30.6	37.3	46.3
53.9	59.8	66.2									

Data Set 18 The following data set represents the number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level, Lawless[10]

15	20	38	42	61	76	86	98	121	146	149	157
175	176	180	180	198	220	224	251	264	282	321	325
653											

Table 1 ML Estimates, -2ln L, K-S Statistics and p-values of the fitted distributions of data sets 1 to 18

	Model	**ML Estimates**			**-2ln L**	**K-S Statistic**	**P-Value**
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$			
Data 1	GLD	17. 1792	14. 3378	11.7653	47.784	0.809	0.000
	GGD	0.6831	7.2644	0.0176	29.238	0.796	0.000
Data 2	GLD	7.0755	0.4492	0.1152	914.950	0.098	0.281
	GGD	4.8293	1.3071	0.0188	912.437	0.087	0.429
Data 3	GLD	3.0404	3.5804	0.0557	226.060	0.123	0.833
	GGD	8.8487	0.6605	0.5387	225.932	0.112	0.908
Data 4	GLD	1.8488	0.0245	0.0114	1249.850	0.957	0.000
	GGD	2.5118	0.9386	0.0159	1250.768	0.954	0.000
Data 5	GLD	1.0932	5.0688	0.0209	788.575	0.439	0.000
	GGD	26.3684	0.2718	7.9396	781.728	0.476	0.000
Data 6	GLD	17.5655	5.1916	0.6181	147.087	0.067	0.999
	GGD	14.9397	1.1094	0.3437	147.092	0.068	0.999
Data 7	GLD	0.0557	5.0640	0.0047	744.975	0.169	0.072
	GGD	5.5727	0.3912	0.7576	741.716	0.142	0.193
Data 8	GLD	0.0524	5.0750	0.0047	564.096	0.150	0.248
	GGD	27.7234	0.1822	11.2554	555.636	0.079	0.921
Data 9	GLD	1.1851	0.0006	0.1287	822.169	0.877	0.000
	GGD	3.8869	0.5139	1.3883	816.852	0.873	0.000
Data 10	GLD	0.8114	0.0007	0.0144	304.348	0.948	0.000
	GGD	6.4942	0.3080	2.1333	302.680	0.933	0.000
Data 11	GLD	1.0628	0.0006	0.5647	110.826	0.936	0.000
	GGD	5.9538	0.3802	5.2747	109.721	0.927	0.000

Table Continued

	Model	ML Estimates			-2ln L	K-S Statistic	P-Value
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$			
Data 12	GLD	2.0093	0.0007	0.2038	634.600	0.043	0.994
	GGD	3.8037	0.7017	0.8028	634.035	0.036	0.999
Data 13	GLD	0.9427	0.0003	0.0081	173.873	0.726	0.000
	GGD	26.6637	0.1736	12.7036	170.488	0.726	0.000
Data 14	GLD	9.6686	0.0029	5.0891	35.637	0.609	0.000
	GGD	51.4619	0.4350	39.4639	34.376	0.600	0.000
Data 15	GLD	17.9881	14.6111	0.6150	208.233	0.135	0.580
	GGD	19.672	0.9814	0.6800	208.225	0.136	0.562
Data 16	GLD	22.7198	4.7710	9.3907	101.959	0.056	0.979
	GGD	3.5861	2.6483	0.3044	100.581	0.044	0.999
Data 17	GLD	1.2025	0.0832	0.0641	128.161	0.095	0.997
	GGD	0.8597	1.4152	0.0068	127.931	0.095	0.997
Data 18	GLD	0.8186	3.9740	0.0101	304.883	0.132	0.769
	GGD	1.9916	0.9426	0.0152	304.928	0.139	0.719

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Conflict of interest

None.

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