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New intelligent eliminating uncertainty from applied models of real-life problems via quantile functions, pivotal quantities and ancillary statistics to find quantum function representing adequate quantified statistical decisions: theory and applications for exponential distribution

Abstract

The technique used here emphasizes pivotal quantities and ancillary statistics relevant for optimization or obtaining prediction limits (or intervals) for anticipated outcomes under parametric uncertainty and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. It does not require the construction of any tables and is applicable whether the experimental data are complete or Type II censored. The exact prediction limits on order statistics associated with sampling from underlying distributions can be found easily and quickly making tables, simulation, Monte-Carlo estimated percentiles, special computer programs, and approximation unnecessary. The proposed analytical methodology is illustrated in terms of the exponential distribution. Applications to other log-location-scale distributions could follow directly.

Keywords: mathematical models, parametric uncertainty, pivotal quantities, ancillary statistics, certainty quantification, eliminating uncertainty, adequate quantified statistical decisions, numerical examples

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Introduction

The technique used here emphasizes pivotal quantities and ancillary statistics relevant for obtaining statistical predictive or confidence decisions for anticipated outcomes of applied stochastic models under parametric uncertainty and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. It does not require the construction of any tables and is applicable whether the experimental data are complete or Type II censored. The proposed technique is based on a probability transformation and pivotal quantity averaging to solve real-life problems in all areas including engineering, science, industry, automation & robotics, business & finance, medicine and biomedicine. The approach used here is a special case of more general considerations applicable whenever the statistical problem is invariant under a group of transformations, which acts transitively on the parameter space.[1-7]

One-parameter exponential distribution

The exponential distribution is applied in a very wide variety of statistical procedures. The mathematics associated with the exponential distribution is often of a simple nature, and so it is possible to obtain explicit formulas in terms of elementary functions. This distribution is one of the most significant and widely used distributions in statistical practice.

Let $\mathbf{X} = (X_1 \leq ... \leq X_m)$ be the first m ordered observations) in a sample of size n from the one-parameter exponential distribution with the probability density function

$$f_\sigma(x) = \sigma^{-1}\exp\left(-\frac{x}{\sigma}\right), \ \sigma > 0, \ x \geq 0, \tag{1}$$

And the probability distribution function

$$F_\sigma(x) = 1 - \exp\left(-\frac{x}{\sigma}\right), \ \ \overline{F}_\sigma(x) = 1 - F_\sigma(x) = \exp\left(-\frac{x}{\sigma}\right), \tag{2}$$

Where σ is the scale parameter. It is assumed that this parameter is unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and X is a random variable. In this case, the likelihood function is given by

$$L(\mu,\sigma)=\prod_{i=1}^{m} f_{\Theta}(x_i)\left(\overline{F}_{\Theta}(x_h)\right)^{n-m}=\frac{1}{\sigma^m}\exp\left(-\left[\sum_{i=1}^{m}x_i+(n-m)x_m\right]\Big/\sigma\right)$$

$$=\frac{1}{\sigma^m}\exp\left(-\sum_{i=1}^{m}\frac{\left(x_i-x_1+x_1\right)}{\sigma}\right)\times\exp\left(-\frac{(n-m)(x_m-x_1+x_1)}{\sigma}\right)$$

$$=\frac{1}{\sigma^m}\exp\left(-\sum_{i=1}^{m}\frac{\left(x_i-x_1\right)}{\sigma}\right)\exp\left(\frac{-mx_1}{\sigma}\right)\times\exp\left(-\frac{(n-m)(x_m-x_1)}{\sigma}\right)\exp\left(\frac{-nx_1+mx_1}{\sigma}\right)$$

$$=\frac{1}{\sigma^m}\exp\left(-\sum_{i=1}^{m}\frac{\left(x_i-x_1\right)}{\sigma}-\frac{(n-m)(x_m-x_1)}{\sigma}\right)\times\exp\left(\frac{-nx_1}{\sigma}\right)$$

$$=\frac{1}{\sigma^{m-1}}\exp\left(-\frac{\sum_{i=1}^{m}(x_i-x_1)+(n-m)(x_m-x_1)}{\sigma}\right)\times \tag{3}$$

$$\frac{1}{\sigma}\exp\left(-\frac{nx_1}{\sigma}\right)=\frac{1}{\sigma^{m-1}}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{ns_1}{\sigma}\right),$$

Where

$$\mathbf{S}=\left(S_1=X_1,\ S_m=\sum_{i=1}^{m}(X_i-X_1)+(n-m)(X_m-X_1)\right) \tag{4}$$

is the complete sufficient statistic for σ. The probability density function of $\mathbf{S}=(S_1,\ S_m)$ is given by

$$\frac{\frac{1}{\sigma^{m-1}}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{ns_1}{\sigma}\right)}{\frac{1}{s_m^{m-2}}\int_0^\infty\left(\frac{s_m}{\sigma}\right)^{m-2}\exp\left(-\frac{s_m}{\sigma}\right)d\left(\frac{s_m}{\sigma}\right)\times\frac{1}{n}\int_0^\infty\exp\left(-\frac{ns_1}{\sigma}\right)d\left(\frac{ns_1}{\sigma}\right)}=\frac{\frac{1}{\sigma^{m-1}}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{ns_1}{\sigma}\right)}{\frac{\Gamma(m-1)}{s_m^{m-2}}\times\frac{1}{n}}$$

$$=\frac{1}{\Gamma(m-1)\sigma^{m-1}}s_m^{m-2}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{n}{\sigma}\exp\left(-\frac{ns_1}{\sigma}\right)=f_\sigma\left(s_m\right)f_\sigma\left(s_1\right), \tag{5}$$

where

$$f_\sigma\left(s_1\right)=\frac{n}{\sigma}\exp\left(-\frac{ns_1)}{\sigma}\right),\quad s_1\geq 0, \tag{6}$$

$$f_\sigma\left(s_m\right)=\frac{1}{\Gamma(m-1)\sigma^{m-1}}s_m^{m-2}\exp\left(-\frac{s_m}{\sigma}\right),\quad s_m\geq 0, \tag{7}$$

$$V_1=\frac{S_1}{\sigma} \tag{8}$$

is the pivotal quantity, the probability density function of which is given by

$$f_1(v_1)=n\exp\left(-nv_1\right),\quad v_1\geq 0, \tag{9}$$

$$V_m=\frac{S_m}{\sigma} \tag{10}$$

is the pivotal quantity, the probability density function of which is given by

$$f_m\left(v_m\right)=\frac{1}{\Gamma(m-1)}v_m^{m-2}\exp\left(-v_m\right),\quad v_m\geq 0. \tag{11}$$

Adequate mathematical model M1 for the one-parameter exponential distribution

Theorem 1. Let us assume that $Y_1 \leq \ldots \leq Y_m$ will be a new (future) random sample from n ordered observations of the one-parameter exponential distribution with a probability density function (pdf) $f_\sigma(y)$, cumulative distribution function (cdf) $F_\sigma(y)$, where σ is the scale parameter. It is assumed that this parameter is unknown. Then the adequate mathematical model $M1$ for a cumulative probability distribution function of the kth order statistic Y_k, $k\in\{1, 2, \ldots, m\}$, to construct one-sided $\gamma-$ content tolerance limit (or two-sided tolerance interval) for Y_k with confidence level β, is given as follows:

$$\left(M1=\int_0^{F_\Theta(y_k)} f_{k,n-k+1}(r)dr\right)=P_\sigma(Y_k\leq y_k\mid n)=\sum_{j=k}^{n}\binom{n}{j}[F_\sigma(y_k)]^j\ [1-F_\sigma(y_k)]^{n-j}, \tag{12}$$

Where

$$f_{k,n-k+1}(r)=\frac{r^{k-1}(1-r)^{(n-k+1)-1}}{\mathrm{B}\left(k,n-k+1\right)}r,\quad 0<r<1, \tag{13}$$

Is the probability density function (pdf) of the beta distribution $(Beta(k,n-k+1))$ with the shape parameters k *and* $n-k+1,\ F_\sigma(y_k)$ represents the generalized pivotal quantity. Then

$$\frac{d}{dy_k}P_\sigma(Y_k\leq y_k\mid n)=\frac{d}{dy_k}\int_0^{F_\sigma(y_k)} f_{k,n-k+1}(r)dr. \tag{14}$$

Proof. It follows from (12) that

$$\frac{d}{dy_k}P_\sigma(Y_k\leq y_k\mid n)=\frac{d}{dy_k}\sum_{j=k}^{n}\binom{n}{j}[F_\sigma(y_k)]^j\ [1-F_\sigma(y_k)]^{n-j}$$

$$=\sum_{j=k}^{n}\binom{n}{j}\frac{d}{dy_k}\left[F_\sigma(y_k)\right]^j[1-F_\sigma(y_k)]^{n-j}=\frac{F_\sigma(y_k)^{k-1}}{\mathrm{B}\left(k,n-k+1\right)}\left(1-F_\sigma(y_k)\right)^{(n-k+1)-1}f_\sigma(y_k) \tag{15}$$

and

$$\frac{d}{dy_k}\int_0^{F_\sigma(y_k)} f_{k,n-k+1}(r)dr=\frac{F_\sigma(y_k)^{k-1}}{\mathrm{B}\left(k,n-k+1\right)}\left(1-F_\sigma(y_k)\right)^{(m-k+1)-1}f_\sigma(y_k). \tag{16}$$

It follows from (15) and (16) that

$$\frac{d}{dy_k}P_\sigma(Y_k\leq y_k\mid n)=\frac{d}{dy_k}\int_0^{F_\sigma(y_k)} f_{k,n-k+1}(r)dr. \tag{17}$$

This ends the proof.

Theorem 2. Let $X_1\leq \ldots \leq X_m$ be the first m ordered observations from the preliminary sample of size n from a one-parameter exponential distribution defined by the probability density function (1). Then a (γ,β) upper one- sided γ-content tolerance limit (with a confidence level β) y_k^U on the kth order statistic Y_k from a set of n future ordered observations $Y_1\leq \ldots \leq Y_n$ also from the distribution (1), which satisfies

$$E\left\{\Pr\left(P_\Theta(Y_k\leq y_k^U\mid n)\geq\gamma\right)\right\}=\beta, \tag{18}$$

is given by

$$y_k^U = \begin{cases} S_1 + \frac{S_m}{n}\left[1 - \left(\frac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}}\right], & \text{if } \left(\frac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}} \le 1, \\ S_1 + \frac{S_m}{n}\left[\left(\frac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}} - 1\right], & \text{if } \left(\frac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}} > 1, \end{cases} \tag{19}$$

where

$$\Omega_\gamma = 1 - q_{(k,n-k+1),\gamma}\left(Beta(k, n-k+1),\ \gamma \text{ quantile}\right). \tag{20}$$

Proof. It follows from (18) and (20) that

$$E\left\{\Pr\left(P_\Theta(Y_k \le y_k^U \mid n) \ge \gamma\right)\right\} = E\left\{\Pr\left(\int_0^{F_\Theta(y_k^U)} f_{k,n-k+1}(r)dr \ge \gamma\right)\right\}$$

$$= E\left\{\Pr\left(1 - \exp\left(-\frac{y_k^U}{\sigma}\right) \ge q_{k,n-k+1;\gamma}\right)\right\} = E\left\{\Pr\left(\exp\left(-\frac{y_k^U}{\sigma}\right) \le 1 - q_{k,n-k+1;\gamma}\right)\right\}$$

$$= E\left\{\Pr\left(-\frac{y_k^U}{\sigma} \le \ln\left(1 - q_{k,n-k+1;\gamma}\right)\right)\right\} = E\left\{\Pr\left(\frac{y_k^U}{\sigma} \ge -\ln\left(1 - q_{k,n-k+1;\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(\frac{y_k^U - S_1}{S_m}\frac{S_m}{\sigma} + \frac{S_1}{\sigma} \ge -\ln\left(1 - q_{k,n-k+1;\gamma}\right)\right)\right\} = E\left\{\Pr\left(\frac{S_1}{\sigma} \ge -\frac{y_k^U - S_1}{S_m}\frac{S_m}{\sigma} - \ln\left(1 - q_{k,n-k+1;\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(V_1 \ge -\eta_k^U V_m - \ln\Omega_\gamma\right)\right\} = E\left\{1 - \Pr\left(V_1 \le -\eta_k^U V_m - \ln\Omega_\gamma\right)\right\} = E\left\{1 - \int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} f_1(v_1)dv_1\right\}, \tag{21}$$

Where

$$\eta_k^U = \frac{y_k^U - S_1}{S_m}. \tag{22}$$

It follows from (21) and (22) that

$$E\left\{1 - \int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} f_1(v_1)dv_1\right\} = E\left\{1 - \int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} n\exp(-nv_1)dv_1\right\} = E\left\{\exp\left(n\eta_k^U V_m\right)\exp\left(\ln\Omega_\gamma^n\right)\right\}$$

$$= E\left\{\Omega_\gamma^n \exp\left(h\eta_k^U V_m\right)\right\} = \int_0^\infty \left(\Omega_\gamma^n \exp\left(n\eta_k^U v_m\right)\right) f_m(v_m)dv_m = \int_0^\infty \left(\Omega_\gamma^n \exp\left(n\eta_k^U v_m\right)\right)\frac{1}{\Gamma(m-1)}v_m^{m-2}\exp\left(-v_m\right)dv_m$$

$$= \Omega_\gamma^n \int_0^\infty \frac{1}{\Gamma(m-1)}v_m^{m-2}\exp\left(-v_m\left[1 - n\eta_k^U\right]\right)dv_m = \frac{\Omega_\gamma^n}{\left[1 - n\eta_k^U\right]^{m-1}} = \beta. \tag{23}$$

It follows from (22) and (23) that

$$\eta_k^U = \frac{y_k^U - S_1}{S_m} = \frac{1}{n}\left(1 - \left[\frac{\Omega_\gamma^n}{\beta}\right]^{\frac{1}{m-1}}\right). \tag{24}$$

It follows from (24) that

$$y_k^U = S_1 + \frac{S_m}{n}\left(1 - \left[\frac{\Omega_\gamma^n}{\beta}\right]^{\frac{1}{m-1}}\right). \tag{25}$$

Then (19) follows from (25). This ends the proof.

Theorem 3. Let $X_1 \le \ldots \le X_m$ be the first m ordered observations from the preliminary sample of size n from the one-parameter exponential distribution defined by the probability density function (1). Then the lower one-sided γ - content tolerance limit (with a confidence level β) y_k^L on the kth order statistic Y_k from a set of n future ordered observations $Y_1 \le \ldots \le Y_n$ also from the distribution (1)), which satisfies

$$E\left\{\Pr\left(P_\mu(Y_k > y_k^L \mid n) \ge \gamma\right)\right\} = \beta, \tag{26}$$

is given by

$$y_k^L = \begin{cases} S_1 + \dfrac{S_m}{n}\left[1 - \left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}}\right], & \text{if } \left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}} \le 1, \\ S_1 + \dfrac{S_m}{n}\left[\left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}} - 1\right], & \text{if } \left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}} > 1, \end{cases} \tag{27}$$

where

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma}\left(Beta(k, n\text{-}k+1), 1-\gamma \text{ quantile}\right). \tag{28}$$

Proof. It follows from (26) and (28) that

$$E\left\{\Pr\left(P_\Theta(Y_k > y_k^L \mid n) \ge \gamma\right)\right\} = E\left\{\Pr\left(\int_0^{F_\Theta(y_k^L)} f_{k,n-k+1}(r)dr \le 1-\gamma\right)\right\}$$
$$= E\left\{\Pr\left(\exp\left(-\frac{y_k^L}{\sigma}\right) \ge 1 - q_{k,n-k+1;1-\gamma}\right)\right\} = E\left\{\Pr\left(\frac{y_k^L - S_1}{S_m}\frac{S_m}{\sigma} + \frac{S_1}{\sigma} \le -\ln\left(1 - q_{k,n-k+1;1-\gamma}\right)\right)\right\}$$
$$= E\left\{\Pr\left(\frac{S_1}{\sigma} \le -\frac{y_k^L - S_1}{S_m}\frac{S_m}{\sigma} - \ln\left(1 - q_{k,n-k+1;1-\gamma}\right)\right)\right\} = E\left\{\Pr\left(V_1 \le -\eta_k^L V_m - \ln\Omega_{1-\gamma}\right)\right\}$$
$$= E\left\{\int_0^{-\eta_k^L V_m - \ln\Omega_{1-\gamma}} f_1(v_1)dv_1\right\}, \tag{29}$$

Where

$$\eta_k^L = \frac{y_k^L - S_1}{S_m}. \tag{30}$$

It follows from (9) and (29) that

$$E\left\{\int_0^{-\eta_k^L V_h - \ln\Omega_{1-\gamma}} f_1(v_1)dv_1\right\} = E\left\{\int_0^{-\eta_k^L V_h - \ln\Omega_{1-\gamma}} n\exp\left(-nv_1\right)dv_1\right\}$$

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$$= E\left\{1-\exp\left(-n\left[-\eta_k^L V_m - \ln\Omega_{1-\gamma}\right]\right)\right\} = E\left\{1-\exp\left(n\eta_k^L V_m\right)\exp\left(n\ln\Omega_{1-\gamma}\right)\right\}$$

$$= E\left\{1-\Omega_{1-\gamma}^n \exp\left(n\eta_k^L V_m\right)\right\} = \int_0^\infty \left(1-\Omega_{1-\gamma}^n \exp\left(n\eta_k^L v_m\right)\right) f_m(v_m)dv_m$$

$$= \int_0^\infty \left(1-\Omega_{1-\gamma}^n \exp\left(n\eta_k^L v_m\right)\right)\frac{1}{\Gamma(m-1)} v_m^{m-2}\exp\left(-v_m\right)dv_m = 1-\Omega_{1-\gamma}^n \int_0^\infty \frac{1}{\Gamma(m-1)} v_m^{m-2}\exp\left(-v_m\left[1-n\eta_k^L\right]\right)dv_m$$

$$= 1-\frac{\Omega_{1-\gamma}^n}{\left[1-n\eta_k^L\right]^{m-1}} = \beta. \tag{31}$$

It follows from (30) and (31) that

$$\eta_{L_k} = \frac{y_k^L - S_1}{S_m} = \frac{1}{n}\left(1-\left[\frac{\Omega_{1-\gamma}^n}{1-\beta}\right]^{\frac{1}{m-1}}\right). \tag{32}$$

It follows from (32) that

$$y_k^L = S_1 + \frac{S_m}{n}\left(1-\left[\frac{\Omega_{1-\gamma}^n}{1-\beta}\right]^{\frac{1}{m-1}}\right). \tag{33}$$

Then (27) follows from (33). This ends the proof.

Numerical practical example

Let us assume that k =5, m =8, n=10, $\gamma = \beta = 0.95$,

$$\mathbf{S} = \left(S_1 = Y_1 = 9,\ S_m = \sum_{i=1}^m (Y_i - Y_1) + (n-m)(Y_m - Y_1)\right)$$

$$= \left(S_1 = 9,\ S_m = 0+1+2+4+6+10+15+23+\ (10-8)\times 23 = 107\right). \tag{34}$$

Then , the $(\gamma = 0.95, \beta = 0.95)$ upper, one-sided $\gamma -$ content tolerance limit y_k^U with confidence level β can be obtained from (25), where the quantile of $Beta(k, n - k + 1), \gamma$ is given by

$$q_{(k,n-k+1),\gamma} = 0.609138, \tag{35}$$

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma} = 1-0.609138 = 0.390862. \tag{36}$$

It follows from (25), (34) and (36) that

$$y_k^U = S_1 + \frac{S_m}{n}\left[1-\left(\frac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}}\right] = 9+\frac{107}{10}\left[\left(1-\frac{\left[0.390862\right]^{10}}{0.95}\right)^{\frac{1}{8-1}}\right] = 9+7.883285 = 16.883285. \tag{37}$$

The $(\gamma = 0.95, \beta = 0.95)$ lower, one-sided $\gamma -$ content tolerance limit y_k^L with confidence level β can be obtained from (33), where the quantile of $Beta(k,n-k+1),1-\gamma$ is given by

$$q_{(k,n-k+1),1-\gamma} = 0.181025 \tag{38}$$

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma} = 1 - 0.181025 = 0.818975. \tag{39}$$

It follows from (27), (34) and (39) that

$$y_k^L = S_1 + \frac{S_m}{n}\left[\left(\frac{\Omega_\gamma^n}{1-\beta}\right)^{\frac{1}{m-1}} - 1\right] = 9 + \frac{107}{10}\left[\left(\frac{[0.818975]^{10}}{1-0.95}\right)^{\frac{1}{8-1}} - 1\right] = 9 + \frac{107}{10}[1.15335326 - 1] = 10.64088. \tag{40}$$

The $(\gamma = 0.95, \beta = 0.95)$ two-sided $\gamma-$ content tolerance interval with confidence level β can be obtained by using (37) and (40):

$$[y_k^L, y_k^U] = [10.64088, 16.883285], \tag{41}$$

where (41) is the certainty quantum (End-to-End).

Two-Parameter Exponential Distribution

Let $\mathbf{X} = (X_1 \le ... \le X_m)$ be the first m ordered observations) in a sample of size n from the two-parameter exponential distribution with the probability density function

$$f_\Theta(x) = \sigma^{-1}\exp\left(-\frac{x-\mu}{\sigma}\right), \quad \mu \ge 0, \ \sigma > 0, \tag{42}$$

and the probability distribution function

$$F_\Theta(x) = 1 - \exp\left(-\frac{x-\mu}{\sigma}\right), \quad \overline{F}_\Theta(x) = 1 - F_\Theta(x) = \exp\left(-\frac{x-\mu}{\sigma}\right), \tag{43}$$

Where $\Theta = (\mu, \sigma)$, μ is the shift parameter and σ is the scale parameter. It is assumed that these parameters are unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and X is a random variable. In this case, the likelihood function is given by

$$L(\mu,\sigma) = \prod_{i=1}^{m} f_\Theta(x_i)\left(\overline{F}_\Theta(x_h)\right)^{n-m} = \frac{1}{\sigma^m}\exp\left(-\left[\sum_{i=1}^{m}(x_i-\mu) + (n-m)(x_m-\mu)\right]\Big/\sigma\right)$$

$$= \frac{1}{\sigma^m}\exp\left(-\sum_{i=1}^{m}\frac{(x_i - x_1 + x_1 - \mu)}{\sigma}\right) \times \exp\left(-\frac{(n-m)(x_m - x_1 + x_1 - \mu)}{\sigma}\right)$$

$$= \frac{1}{\sigma^m}\exp\left(-\sum_{i=1}^{m}\frac{(x_i - x_1)}{\sigma}\right)\exp\left(-\frac{m(x_1-\mu)}{\sigma}\right) \times \exp\left(-\frac{(n-m)(x_m - x_1)}{\sigma}\right)\exp\left(-\frac{(n-m)(x_1-\mu)}{\sigma}\right)$$

$$= \frac{1}{\sigma^{m-1}}\exp\left(-\frac{\sum_{i=1}^{m}(x_i - x_1) + (n-m)(x_m - x_1)}{\sigma}\right) \times \frac{1}{\sigma}\exp\left(-\frac{n(x_1-\mu)}{\sigma}\right)$$

$$=\frac{1}{\sigma^{m-1}}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right), \tag{44}$$

where

$$\mathbf{S}=\left(S_1=X_1,\ S_m=\sum_{i=1}^{m}(X_i-X_1)+(n-m)(X_m-X_1)\right) \tag{45}$$

is the complete sufficient statistic for $\Theta=(\mu,\sigma)$. The probability density function of $\mathbf{S}=(S_1, S_m)$ is given by

$$\frac{\frac{1}{\sigma^{m-1}}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)}{\frac{1}{s_m^{m-2}}\int_0^\infty\left(\frac{s_m}{\sigma}\right)^{m-2}\exp\left(-\frac{s_m}{\sigma}\right)d\left(\frac{s_m}{\sigma}\right)\times\frac{1}{n}\int_0^\infty\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)d\left(\frac{n(s_1-\mu)}{\sigma}\right)}=\frac{\frac{1}{\sigma^{m-1}}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)}{\frac{\Gamma(m-1)}{s_m^{m-2}}\times\frac{1}{n}}$$

$$=\frac{1}{\Gamma(m-1)\sigma^{m-1}}s_m^{m-2}\exp\left(-\frac{s_m}{\sigma}\right)\times\frac{n}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)=f_\sigma(s_m)f_\Theta(s_1), \tag{46}$$

where

$$f_\Theta(s_1)=\frac{n}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right),\quad s_1\ge\mu, \tag{47}$$

$$f_\sigma(s_m)=\frac{1}{\Gamma(m-1)\sigma^{m-1}}s_m^{m-2}\exp\left(-\frac{s_m}{\sigma}\right),\quad s_m\ge 0, \tag{48}$$

$$V_1=\frac{S_1-\mu}{\sigma} \tag{49}$$

is the pivotal quantity, the probability density function of which is given by

$$f_1(v_1)=n\exp(-nv_1),\quad v_1\ge 0, \tag{50}$$

$$V_m=\frac{S_m}{\sigma} \tag{51}$$

is the pivotal quantity, the probability density function of which is given by

$$f_m(v_m)=\frac{1}{\Gamma(m-1)}v_m^{m-2}\exp(-v_m),\quad v_m\ge 0. \tag{52}$$

Adequate mathematical model M2 for the two-parameter exponential distribution

Theorem 3. Let us assume that $Y_1\le\ldots\le Y_m$ will be a new (future) random sample from n ordered observations of the exponential distribution with a probability density function (pdf) $f_\Theta(y)$, cumulative distribution function (cdf) $F_\Theta(y)$, where $\Theta=(\mu,\sigma)$, μ is the shift parameter and σ is the scale parameter. It is assumed that these parameters are unknown. Then the adequate mathematical model $M2$ for a cumulative probability distribution function of the kth order statistic Y_k, $k\in\{1, 2, \ldots, m\}$, to construct one-sided γ − content tolerance limit (or two-sided tolerance interval) for Y_k with confidence level β, is given as follows:

$$\left(M2=\int_0^{F_\Theta(y_k)}f_{k,n-k+1}(r)dr\right)=P_\Theta(Y_k\le y_k\,|\,n)=\sum_{j=k}^{n}\binom{n}{j}[F_\Theta(y_k)]^j\ [1-F_\Theta(y_k)]^{n-j}, \tag{53}$$

Where

$$f_{k,n-k+1}(r)=\frac{r^{k-1}(1-r)^{(n-k+1)-1}}{\mathrm{B}\left(k,n-k+1\right)}r,\quad 0<r<1, \tag{54}$$

Is the probability density function (pdf) of the beta distribution $(Beta(k,n-k+1))$ with the shape parameters k *and* $n-k+1,$ $F_{\Theta}(y_k)$ represents the generalized pivotal quantity. Then

$$\frac{d}{dy_k}P_{\Theta}(Y_k\le y_k\,|\,n)=\frac{d}{dy_k}\int_0^{F_{\Theta}(y_k)}f_{k,n-k+1}(r)dr. \tag{55}$$

Proof. It follows from (55) that

$$\frac{d}{dy_k}P_{\Theta}(Y_k\le y_k\,|\,n)=\frac{d}{dy_k}\sum_{j=k}^{n}\binom{n}{j}[F_{\Theta}(y_k)]^j\;[1-F_{\Theta}(y_k)]^{n-j}$$

$$=\sum_{j=k}^{n}\binom{n}{j}\frac{d}{dy_k}\left[F_{\Theta}(y_k)\right]^j[1-F_{\Theta}(y_k)]^{n-j}=\frac{F_{\Theta}(y_k)^{k-1}}{\mathrm{B}\left(k,n-k+1\right)}\left(1-F_{\Theta}(y_k)\right)^{(n-k+1)-1}f_{\Theta}(y_k) \tag{56}$$

and

$$\frac{d}{dy_k}\int_0^{F_{\Theta}(y_k)}f_{k,n-k+1}(r)dr=\frac{F_{\Theta}(y_k)^{k-1}}{\mathrm{B}\left(k,n-k+1\right)}\left(1-F_{\Theta}(y_k)\right)^{(m-k+1)-1}f_{\Theta}(y_k). \tag{57}$$

It follows from (56) and (57) that

$$\frac{d}{dy_k}P_{\Theta}(Y_k\le y_k\,|\,n)=\frac{d}{dy_k}\int_0^{F_{\Theta}(y_k)}f_{k,n-k+1}(r)dr. \tag{58}$$

This ends the proof.

Theorem 4. Let $X_1\le\ldots\le X_m$ be the first m ordered observations from the preliminary sample of size n from a two-parameter exponential distribution defined by the probability density function (42). Then a (γ,β) upper one- sided γ-content tolerance limit (with a confidence level β) y_k^U on the kth order statistic Y_k from a set of n future ordered observations $Y_1\le\ldots\le Y_n$ also from the distribution (42), which satisfies

$$E\left\{\Pr\left(P_{\Theta}(Y_k\le y_k^U\,|\,n)\ge\gamma\right)\right\}=\beta, \tag{59}$$

is given by

$$y_k^U=\begin{cases}S_1+\dfrac{S_m}{n}\left[1-\left(\dfrac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}}\right], & \text{if } \left(\dfrac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}}\le 1,\\[2ex] S_1+\dfrac{S_m}{n}\left[\left(\dfrac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}}-1\right], & \text{if } \left(\dfrac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}}>1,\end{cases} \tag{60}$$

Where

$$\Omega_\gamma=1-q_{(k,n-k+1),\gamma}\left(Beta(k,n\text{-}k+1),\,\gamma\ \text{ quantile}\right). \tag{61}$$

Proof. It follows from (59) that

$$E\left\{\Pr\left(P_{\Theta}(Y_k \le y_k^U \mid n) \ge \gamma\right)\right\} = E\left\{\Pr\left(\int_0^{F_{\Theta}(y_k^U)} f_{k,n-k+1}(r)dr \ge \gamma\right)\right\}$$

$$= E\left\{\Pr\left(1-\exp\left(-\frac{y_k^U-\mu}{\sigma}\right) \ge q_{k,n-k+1;\gamma}\right)\right\} = E\left\{\Pr\left(\exp\left(-\frac{y_k^U-\mu}{\sigma}\right) \le 1-q_{k,n-k+1;\gamma}\right)\right\}$$

$$= E\left\{\Pr\left(-\frac{y_k^U-\mu}{\sigma} \le \ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\} = E\left\{\Pr\left(\frac{y_k^U-\mu}{\sigma} \ge -\ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(\frac{y_k^U-S_1}{S_m}\frac{S_m}{\sigma} + \frac{S_1-\mu}{\sigma} \ge -\ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\} = E\left\{\Pr\left(\frac{S_1-\mu}{\sigma} \ge -\frac{y_k^U-S_1}{S_m}\frac{S_m}{\sigma} - \ln\left(1-q_{k,n-k+1;\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(V_1 \ge -\eta_k^U V_m - \ln\Omega_\gamma\right)\right\} = E\left\{1-\Pr\left(V_1 \le -\eta_k^U V_m - \ln\Omega_\gamma\right)\right\} = E\left\{1-\int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} f_1(v_1)dv_1\right\}, \tag{62}$$

where

$$\eta_k^U = \frac{y_k^U - S_1}{S_m}. \tag{63}$$

It follows from (62) and (63) that

$$E\left\{1-\int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} f_1(v_1)dv_1\right\} = E\left\{1-\int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} n\exp\left(-nv_1\right)dv_1\right\}$$

$$= E\left\{1-\int_0^{-\eta_k^U V_m - \ln\Omega_\gamma} n\exp\left(-nv_1\right)dv_1\right\} = E\left\{\exp\left(n\eta_k^U V_m\right)\exp\left(\ln\Omega_\gamma^n\right)\right\}$$

$$= E\left\{\Omega_\gamma^n \exp\left(h\eta_k^U V_m\right)\right\} = \int_0^\infty \left(\Omega_\gamma^n \exp\left(n\eta_k^U v_m\right)\right) f_m(v_m)dv_m = \int_0^\infty \left(\Omega_\gamma^n \exp\left(n\eta_k^U v_m\right)\right)\frac{1}{\Gamma(m-1)} v_m^{m-2}\exp\left(-v_m\right)dv_m$$

$$= \Omega_\gamma^n \int_0^\infty \frac{1}{\Gamma(m-1)} v_m^{m-2} \exp\left(-v_m\left[1-n\eta_k^U\right]\right)dv_m = \frac{\Omega_\gamma^n}{\left[1-n\eta_k^U\right]^{m-1}} = \beta. \tag{64}$$

It follows from (63) and (64) that

(65)

It follows from (65) that

$$y_k^U = S_1 + \frac{S_m}{n}\left(1-\left[\frac{\Omega_\gamma^n}{\beta}\right]^{\frac{1}{m-1}}\right). \tag{66}$$

Then (60) follows from (66). This ends the proof.

Theorem 5. Let $X_1 \leq \ldots \leq X_m$ be the first m ordered observations from the preliminary sample of size n from a two-parameter exponential distribution defined by the probability density function (42). Then the lower one-sided γ - content tolerance limit (with a confidence level β) y_k^L on the kth order statistic Y_k from a set of n future ordered observations $Y_1 \leq \ldots \leq Y_n$ also from the distribution (42)), which satisfies

$$E\left\{\Pr\left(P_\mu(Y_k > y_k^L \mid n) \geq \gamma\right)\right\} = \beta, \tag{67}$$

is given by

$$y_k^L = \begin{cases} S_1 + \dfrac{S_m}{n}\left[1 - \left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}}\right], & \text{if } \left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}} \leq 1, \\ S_1 + \dfrac{S_m}{n}\left[\left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}} - 1\right], & \text{if } \left(\dfrac{\Omega_{1-\gamma}^n}{1-\beta}\right)^{\frac{1}{m-1}} > 1, \end{cases} \tag{68}$$

Where

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma}\left(Beta(k, n-k+1), 1-\gamma \text{ quantile}\right). \tag{69}$$

Proof. It follows from (67) and (69) that

$$E\left\{\Pr\left(P_\Theta(Y_k > y_k^L \mid n) \geq \gamma\right)\right\} = E\left\{\Pr\left(\int_0^{F_\Theta(y_k^L)} f_{k,n-k+1}(r)dr \leq 1-\gamma\right)\right\}$$

$$= E\left\{\Pr\left(\exp\left(-\frac{y_k^L - \mu}{\sigma}\right) \geq 1 - q_{k,n-k+1;1-\gamma}\right)\right\} = E\left\{\Pr\left(\frac{y_k^L - S_1}{S_m}\frac{S_m}{\sigma} + \frac{S_1 - \mu}{\sigma} \leq -\ln\left(1 - q_{k,n-k+1;1-\gamma}\right)\right)\right\}$$

$$= E\left\{\Pr\left(\frac{S_1 - \mu}{\sigma} \leq -\frac{y_k^L - S_1}{S_m}\frac{S_m}{\sigma} - \ln\left(1 - q_{k,n-k+1;1-\gamma}\right)\right)\right\} = E\left\{\Pr\left(V_1 \leq -\eta_k^L V_m - \ln \Omega_{1-\gamma}\right)\right\}$$

$$= E\left\{\int_0^{-\eta_k^L V_m - \ln \Omega_{1-\gamma}} f_1(v_1)dv_1\right\}, \tag{70}$$

where

$$\eta_k^L = \frac{y_k^L - S_1}{S_m}. \tag{71}$$

It follows from (50) and (70) that

$$E\left\{\int_0^{-\eta_k^L V_h - \ln \Omega_{1-\gamma}} f_1(v_1)dv_1\right\} = E\left\{\int_0^{-\eta_k^L V_h - \ln \Omega_{1-\gamma}} n\exp\left(-nv_1\right)dv_1\right\}$$

$$= E\left\{1 - \exp\left(-n\left[-\eta_k^L V_m - \ln \Omega_{1-\gamma}\right]\right)\right\} = E\left\{1 - \exp\left(n\eta_k^L V_m\right)\exp\left(q \ln \Omega_{1-\gamma}\right)\right\}$$

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$$= E\left\{1-\Omega_{1-\gamma}^{n}\exp\left(n\eta_k^L V_m\right)\right\} = \int_0^\infty \left(1-\Omega_{1-\gamma}^{n}\exp\left(n\eta_k^L v_m\right)\right) f_m(v_m)dv_m$$

$$= \int_0^\infty \left(1-\Omega_{1-\gamma}^{n}\exp\left(n\eta_k^L v_m\right)\right)\frac{1}{\Gamma(m-1)}v_m^{m-2}\exp\left(-v_m\right)dv_m = 1-\Omega_{1-\gamma}^{n}\int_0^\infty \frac{1}{\Gamma(m-1)}v_m^{m-2}\exp\left(-v_m\left[1-n\eta_k^L\right]\right)dv_m$$

$$= 1-\frac{\Omega_{1-\gamma}^{n}}{\left[1-n\eta_k^L\right]^{m-1}} = \beta. \quad (72)$$

It follows from (71) and (72) that

$$\eta_{L_k} = \frac{y_k^L - S_1}{S_m} = \frac{1}{n}\left(1-\left[\frac{\Omega_{1-\gamma}^{n}}{1-\beta}\right]^{\frac{1}{m-1}}\right). \quad (73)$$

It follows from (73) that

$$y_k^L = S_1 + \frac{S_m}{n}\left(1-\left[\frac{\Omega_{1-\gamma}^{n}}{1-\beta}\right]^{\frac{1}{m-1}}\right). \quad (74)$$

Then (68) follows from (74). This ends the proof.

Numerical practical example

Let us assume that k =5, m =8, n=10, $\gamma = \beta = 0.95$,

$$\mathbf{S} = \left(S_1 = Y_1 = 9,\ S_m = \sum_{i=1}^{m}(Y_i - Y_1) + (n-m)(Y_m - Y_1)\right)$$

$$= \left(S_1 = 9,\ S_m = 0+1+2+4+6+10+15+23+\ (10-8)\times 23 = 107\right). \quad (75)$$

Then , the $(\gamma = 0.95, \beta = 0.95)$ upper, one-sided $\gamma -$ content tolerance limit y_k^U with confidence level β can be obtained from (61), where the quantile of $Beta(k, n-k+1), \gamma$ is given by

$$q_{(k,n-k+1),\gamma} = 0.609138, \quad (76)$$

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma} = 1-0.609138 = 0.390862. \quad (77)$$

It follows from (60), (75) and (77) that

$$y_k^U = S_1 + \frac{S_m}{n}\left[1-\left(\frac{\Omega_\gamma^n}{\beta}\right)^{\frac{1}{m-1}}\right] = 9 + \frac{107}{10}\left[\left(1-\frac{\left[0.390862\right]^{10}}{0.95}\right)^{\frac{1}{8-1}}\right] = 9 + 7.883285 = 16.883285. \quad (78)$$

The $(\gamma = 0.95, \beta = 0.95)$ lower, one-sided $\gamma -$ content tolerance limit y_k^L with confidence level β can be obtained from (68), where the quantile of $Beta(k, n-k+1), 1-\gamma$ is given by

$$q_{(k,n-k+1),1-\gamma} = 0.181025 \tag{79}$$

$$\Omega_{1-\gamma} = 1 - q_{(k,n-k+1),1-\gamma} = 1 - 0.181025 = 0.818975. \tag{80}$$

It follows from (68), (75) and (80) that

$$y_k^L = S_1 + \frac{S_m}{n}\left[\left(\frac{\Omega_\gamma^h}{1-\beta}\right)^{\frac{1}{m-1}} - 1\right] = 9 + \frac{107}{10}\left[\left(\frac{[0.818975]^{10}}{1-0.95}\right)^{\frac{1}{8-1}} - 1\right] = 9 + \frac{107}{10}[1.15335326 - 1] = 10.64088. \tag{81}$$

The $(\gamma = 0.95, \beta = 0.95)$ two-sided $\gamma -$ content tolerance interval with confidence level β can be obtained by using (78) and (81):

$$\left[y_k^L, y_k^U\right] = [10.64088, 16.883285], \tag{82}$$

where (82) is the certainty quantum (End-to-End).

Conclusion

The novel unified technique of computational intelligence proposed in this paper represents the conceptually simple, efficient and useful approach to constructing exact, optimal or improved statistical decision rules under parametric uncertainty of applied stochastic models. The technique is based on the constructive use of the invariance principle in mathematical statistics. We have illustrated the technique using the exponential distribution. Applications to other log-location-scale distributions could follow directly. The methodology described here can be extended in several different directions to solve various problems arising in practice.

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Conflicts of interest

The authors declare that there are no conflicts of interest.

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