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Novel constructing adequate simultaneous predictive limits or confidence intervals for future outcomes via pivotal quantities and ancillary statistics in the case of parametric uncertainty of applied real-life models

Abstract

In this paper, we consider the problems of constructing simultaneous predictive limits on future outcomes of all of l future samples using the results of a previous sample from the same underlying distribution belonging to invariant family. The approach used here emphasizes pivotal quantities relevant for obtaining ancillary statistics and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. It does not require the construction of any tables and is applicable whether the data are complete or Type II censored. The lower simultaneous predictive limits are often used as warranty criteria by manufacturers. The technique used here emphasizes pivotal quantities relevant for obtaining ancillary statistics and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. Applications of the proposed procedures are given for the two-parameter exponential distribution. The proposed technique is based on a probability transformation and pivotal quantity averaging to solve real-life problems in all areas including engineering, science, industry, automation & robotics, business & finance, medicine and biomedicine. It is conceptually simple and easy to use. The exact lower simultaneous predictive limits are found and illustrated with a numerical example.

Keywords: future samples, observations, exact lower simultaneous predictive limits, statistical methods of constructing

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Introduction

Simultaneous predictive limits are required in many practical applications. In particular, it is often necessary to construct lower simultaneous predictive limits that are exceeded with probability $1-\alpha$ by observations or functions of observations of all of l future samples, each consisting of m units. The predictive limits depend upon a previously available complete or type II censored sample of size n from the same distribution. For instance, a situation where such simultaneous predictive limits are required is given below:

A customer has placed an order for a product which has an underlying time-to-failure distribution. The terms of his purchase call for l monthly shipments. From each shipment the customer will select a random sample of m units and accept the shipment only if the smallest time to failure for this sample exceeds a specified lower limit. The manufacturer wishes to use the results of a previous sample of n units to calculate this limit so that the probability is $1-\alpha$ that all l shipments will be accepted. It is assumed that the n past units and the lm future units are random samples from the same population.

In this paper we consider lower simultaneous predictive limits. The lower simultaneous predictive limit is based on observations in an initial sample. The technique used here emphasizes pivotal quantities relevant for obtaining ancillary statistics.[1–7] The exact lower simultaneous predictive limit on future order statistics is obtained via the technique of invariant embedding and illustrated with numerical example.

Two-parameter exponential distribution

Let $\mathbf{X} = (X_1 \le \ldots \le X_k)$ be the first k ordered observations (order statistics) in a sample of size n from the two-parameter exponential distribution with the probability density function

$$f_\vartheta(x) = \sigma^{-1}\exp\left(-\frac{x-\mu}{\sigma}\right), \quad \sigma > 0,\ \mu \ge 0, \tag{1}$$

and the cumulative probability distribution function

$$F_\vartheta(x) = 1-\exp\left(-\frac{x-\mu}{\sigma}\right),$$

$$\bar{F}_\vartheta(x) = 1-F_\vartheta(x) = \exp\left(-\frac{x-\mu}{\sigma}\right), \tag{2}$$

where $\vartheta = (\mu,\sigma)$, μ is the shift parameter and σ is the scale parameter. It is assumed that these parameters are unknown. In Type II censoring, which is of primary interest here, the number of survivors is fixed and X is a random variable. In this case, the likelihood function is given by

$$L(\mu,\sigma) = \prod_{i=1}^{k} f_\vartheta(x_i)\left(\bar{F}_\vartheta(x_k)\right)^{n-k}$$

$$= \frac{1}{\sigma^k}\exp\left(-\left[\sum_{i=1}^{k}(x_i-\mu) + (n-k)(x_k-\mu)\right]\Big/\sigma\right)$$

$$=\frac{1}{\sigma^k}\exp\left(-\left[\sum_{i=1}^{k}(x_i-x_1+x_1-\mu)+(n-k)(x_k-x_1+x_1-\mu)\right]\Big/\sigma\right)$$

$$=\frac{1}{\sigma^{k-1}}\exp\left(-\left[\sum_{i=1}^{k}(x_i-x_1)+(n-k)(x_k-x_1)\right]\Big/\sigma\right)$$

$$\times\frac{1}{\sigma}\exp\left(-\frac{n(x_1-\mu)}{\sigma}\right)$$

$$=\frac{1}{\sigma^{k-1}}\exp\left(-\frac{s_k}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{k(s_1-\mu)}{\sigma}\right), \quad (3)$$

where

$$\mathbf{S}=\left(S_1=X_1,\ S_k=\sum_{i=1}^{k}(Y_i-Y_1)+(n-k)(X_k-X_1)\right) \quad (4)$$

is the complete sufficient statistic for $\vartheta=(\mu,\sigma)$, . The probability density function of $\mathbf{S}=(S_1, S_k)$ is given by

$$f_\vartheta(s_1,s_k)$$

$$=\frac{\frac{1}{\sigma^{k-1}}\exp\left(-\frac{s_k}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)}{\frac{1}{s_k^{k-2}}\int_0^\infty\frac{s_k^{k-2}}{\sigma^{k-1}}\exp\left(-\frac{s_k}{\sigma}\right)ds_k\times\frac{1}{n}\int_0^\infty\frac{n}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)ds_1}$$

$$=\frac{\frac{1}{\sigma^{k-1}}\exp\left(-\frac{s_k}{\sigma}\right)\times\frac{1}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)}{\frac{\Gamma(k-1)}{s_k^{k-2}}\times\frac{1}{n}}$$

$$=\frac{1}{\Gamma(k-1)\sigma^{k-1}}s_k^{k-2}\exp\left(-\frac{s_k}{\sigma}\right)$$

$$\times\frac{n}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)=f_\sigma(s_k)f_\vartheta(s_1), \quad (5)$$

where

$$f_\vartheta(s_1)=\frac{n}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right),\quad s_1\ge\mu, \quad (6)$$

$$f_\sigma(s_k)=\frac{1}{\Gamma(k-1)\sigma^{k-1}}s_k^{k-2}\exp\left(-\frac{s_k}{\sigma}\right),\quad s_k\ge 0, \quad (7)$$

$$V_1=\frac{S_1-\mu}{\sigma} \quad (8)$$

is the pivotal quantity, the probability density function of which is given by

$$f_1(v_1)=n\exp(-nv_1),\quad v_1\ge 0, \quad (9)$$

$$V_k=\frac{S_k}{\sigma} \quad (10)$$

is the pivotal quantity, the probability density function of which is given by

$$f_k(v_k)=\frac{1}{\Gamma(k-1)}v_k^{k-2}\exp(-v_k),\quad v_k\ge 0. \quad (11)$$

Practical example for constructing lower simultaneous prediction limit

Let's assume that an airline has a policy of replacing a specific device used in multiple locations in its fleet avionics systems every 7 months. The airline doesn't want one of these devices to fail before it can be replaced. Shipments of a batch of devices are carried out from each of l enterprises. Each enterprise selects a random sample of m devices. The manufacturer wishes to take this total random sample and calculate the lower limit of simultaneous forecasting such that all deliveries will be accepted with probability $1-\alpha$.

Innovative technique of constructing lower simultaneous prediction limit

For instance, suppose that $X_1\le\ldots\le X_n$ and $Y_{1j}\le\ldots\le Y_{mj}, j\in\{1,\ldots,l\}$, denote $n+lm$ independent and identically distributed random variables from a two –parameter exponential distribution (14), where $\vartheta=(\mu,\sigma)$, μ is the shift parameter and σ is the scale parameter. It is assumed that these parameters are unknown. Let

$$\mathbf{S}=\left(S_1=X_1,\ S_k=\sum_{i=1}^{k}(X_i-X_1)+(n-k)(X_k-X_1)\right), \quad (12)$$

where X_1=8, n=20, l=3, m=5, k=16 and S_k=103.5402, with

$$f_\vartheta(s_1)=\frac{n}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right),\quad s_1\ge\mu, \quad (13)$$

$$f_\sigma(s_k)=\frac{1}{\Gamma(k-1)\sigma^{k-1}}s_k^{k-2}\exp\left(-\frac{s_k}{\sigma}\right),\quad s_k\ge 0. \quad (14)$$

$$f_\vartheta(y_1)=\frac{lm}{\sigma}\exp\left(-\frac{lm(y_1-\mu)}{\sigma}\right),\quad y_1\ge\mu,$$

$$F_\vartheta(y_1)=1-\exp\left(-\frac{lm(y_1-\mu)}{\sigma}\right),$$

$$\overline{F}_\vartheta(y_1)=\exp\left(-\frac{lm(y_1-\mu)}{\sigma}\right). \quad (15)$$

It follows from (15) that

$$\overline{F}_\vartheta(y_1)=\exp\left(-\frac{lm(y_1-\mu)}{\sigma}\right)=\exp\left(-\frac{lm(y_1-s_1+s_1-\mu)}{\sigma}\right)$$

$$=\exp\left(-\frac{lm(y_1-s_1)}{\sigma}\right)\exp\left(-\frac{lm(s_1-\mu)}{\sigma}\right). \quad (16)$$

It follows from (13) and (16) that

$$\int_\mu^\infty\overline{F}_\vartheta(y_1)f_\vartheta(s_1)ds_1=\int_\mu^\infty\exp\left(-\frac{lm(y_1-s_1)}{\sigma}\right)\exp\left(-\frac{lm(s_1-\mu)}{\sigma}\right)\frac{n}{\sigma}\exp\left(-\frac{n(s_1-\mu)}{\sigma}\right)ds_1$$

$$=\frac{n}{n+lm}\exp\left(-\frac{lm(y_1-s_1)}{\sigma}\right)\int_\mu^\infty\frac{n+lm}{\sigma}\exp\left(-\frac{[n+lm](s_1-\mu)}{\sigma}\right)ds_1=\frac{n}{n+lm}\exp\left(-\frac{lm(y_1-s_1)}{\sigma}\right). \quad (17)$$

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It follows from (17) that

$$\frac{n}{n+lm}\exp\left(-\frac{lm(y_1-s_1)}{\sigma}\right)=\frac{n}{n+lm}\exp\left(-\frac{lm(y_1-s_1)}{s_k}\frac{s_k}{\sigma}\right). \quad (18)$$

It follows from (14) and (18) that

$$\int_0^\infty \frac{n}{n+lm}\exp\left(-\frac{lm(y_1-s_1)}{s_k}\frac{s_k}{\sigma}\right)f_\sigma(s_k)ds_k=\int_0^\infty \frac{n}{n+lm}\exp\left(-\frac{lm(y_1-s_1)}{s_k}\frac{s_k}{\sigma}\right)\frac{1}{\Gamma(k-1)\sigma^{k-1}}s_k^{k-2}\exp\left(-\frac{s_k}{\sigma}\right)ds_k$$

$$=\frac{n}{n+lm}\left(1+lm\frac{y_1-s_1}{s_k}\right)^{-(k-1)}, \quad (19)$$

It follows from (16), (17), (18) and (19) that

$$E\{\bar{F}_\vartheta(y_1)\}=\frac{n}{n+lm}\left(1+lm\frac{y_1-s_1}{s_k}\right)^{-(k-1)}=\bar{F}\left(\frac{y_1-s_1}{s_k}\right)=1-F\left(\frac{y_1-s_1}{s_k}\right). \quad (20)$$

If $(1-\alpha)=0.95$, the manufacturer finds from

$$\bar{F}\left(\frac{y_1-s_1}{s_k}\right)=\frac{n}{n+lm}\left(1+lm\frac{y_1-s_1}{s_k}\right)^{-(k-1)}=1-\alpha \quad (21)$$

that

$$y_1=s_1+\frac{s_k}{lm}\left[\left(\frac{n}{(1-\alpha)(n+lm)}\right)^{1/(k-1)}-1\right]=8+\frac{103.5402}{3\times 5}\left[\left(\frac{20}{(1-0.05)(20+3\times 5)}\right)^{1/(16-1)}-1\right]$$

$$=8+6.902679484[-0.03332039]=8-0.23=7.77 \quad (22)$$

and he has 95% assurance that no failures will occur in each shipment before $y_1=7.77$ month intervals.

Constructing confidence interval of equal tails or shortest length for lower simultaneous predictive limit

It follows from (20) that

$$F\left(\frac{y_1-s_1}{s_k}\right)=1-\frac{n}{n+lm}\left(1+lm\frac{y_1-s_1}{s_k}\right)^{-(k-1)}. \quad (23)$$

Let us assume that

$$F\left(\frac{y_{1(2)}-s_1}{s_k}\right)=1-\frac{n}{n+lm}\left(1+lm\frac{y_{1(2)}-s_1}{s_k}\right)^{-(k-1)}=1-\alpha+p \quad (24)$$

and

$$F\left(\frac{y_{1(1)}-s_1}{s_k}\right)=1-\frac{n}{n+lm}\left(1+lm\frac{y_{1(1)}-s_1}{s_k}\right)^{-(k-1)}=p, \quad (25)$$

then

$$F\left(\frac{y_{1(2)}-s_1}{s_k}\right)-F\left(\frac{y_{1(1)}-s_1}{s_k}\right)=1-\alpha+p-p=1-\alpha. \quad (26)$$

It follows from (24) that

$$y_{1(2)}=s_1+\frac{s_k}{lm}\left(\left[\frac{n}{(\alpha-p)(n+lm)}\right]^{1/(k-1)}-1\right). \quad (27)$$

It follows from (25) that

$$y_{1(1)}=s_1+\frac{s_k}{lm}\left(\left[\frac{n}{(1-p)(n+lm)}\right]^{1/(k-1)}-1\right). \quad (28)$$

If p=0.025, $\alpha=0.05$, then the $(1-\alpha)-$ confidence interval of equal tails for y_1 is given by

$$\left[y_{1(1)}=7.758455,\ y_{1(2)}=9.601241\right],\ \left(y_{1(2)}-y_{1(1)}\right)=1.842786. \quad (29)$$

If p=0, $\alpha=0.05$, then the $(1-\alpha)-$ confidence interval of shortest length for y_1 is given by

$$\left[y_{1(1)}=7.747221,\ y_{1(2)}=9.217217\right],\ \left(y_{1(2)}-y_{1(1)}\right)=1.469996. \quad (30)$$

Conclusion

In this paper we propose the novel technique of constructing simultaneous predictive limits on observations or functions of observations in all of k future samples under parametric uncertainty of the underlying distribution. The exact predictive limits are found and illustrated with a numerical example. We have illustrated the proposed methodology for the two-parameter exponential distribution. Application to other distributions could follow directly.

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Conflicts of interest

The authors declare that there is no conflict of interest.

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