

The unified sub-equation method and its applications to conformable space-time fractional fourth-order pochhammer-chree equation

Abstract

In this article, we apply the unified sub-equation method proposed by Lu Bin and Zhang Hong Qing to construct many new Jacobi elliptic function solutions, solitons and other solutions for the conformable space-time fractional fourth-order Pochhammer-Chree equation. This method is direct and more powerful than the projective Riccati equation method. The solitons and other solutions of this equation can be found from the Jacobi elliptic solutions when its modulus $m \rightarrow 1$ or $m \rightarrow 0$ respectively. Comparing our new results with the well-known results is given.

Keywords: unified sub-equation method, jacobi elliptic function solutions, dark, singular and bright solitons, periodic solutions, the conformable space-time fractional fourth-order pochhammer-chree equation

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Zayed EME, Shohib RMA

Mathematics Department, Zagazig University, Egypt

Correspondence: Elsayed ME Zayed, Department of Mathematics, Faculty of Sciences, Zagazig University, Zagazig, Egypt, Email eme_zayad@yahoo.com

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Introduction

When the nonlinear partial differential equations (PDEs) are analyzed, one of the most important equation is the construction of the exact solutions of those equations. Searching for the exact solutions of those equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, geochemistry, thermodynamics, solid mechanics, civil engineering, and non-Newtonian fluids to the natural sciences including population acology, infectious disease epidemiology, natural networks and so on. Throughout the past few decades, a particular attention has been given to the problem of finding the exact solutions of these nonlinear PDEs. By virtue of these solutions, one may give better insight into the physical aspects of the nonlinear models studies. In recent years, quite a few methods for constructing explicit and solitary wave solutions of the nonlinear PDEs have been presented. A variety of powerful methods, such as the inverse scattering method,¹ the Hirota method,² the Bäcklund transform method,^{3,4} the Painlevé expansion method,⁵ the exp-function method,^{6,7} the sub-ODE method,⁸⁻¹⁰ the Jacobi elliptic function method,^{11,12} the sine-cosine function method,^{13,14} the (G'/G) -expansion method,¹⁵⁻¹⁷ the modified simple equation method,^{18,19} the Kudryashov method,^{20,21} the multiple exp-function method,^{22,23} the homogeneous balance method,²⁴ the auxiliary equation method,^{25,26} the extended auxiliary equation method,²⁷⁻²⁹ the soliton ansatz method,³⁰⁻³³ the new mapping method,^{34,35} the first integral method,^{36,37} the $(G'/G, 1/G)$ -expansion method^{38,39} and an unified sub-equation method,⁴⁰ the projective Riccati equation method^{41,42} and so on. Motivated by the projective Riccati equation method proposed by Conte & Musette⁴³ and developed by Yan⁴⁴, we present a new direct algebraic method proposed by Bin L & Hong-Qing Z⁴⁰ to obtain many new double periodic wave solutions of nonlinear PDEs which cannot be acquired by using the projective Riccati equation method.

The objective of this article is to apply a unified sub-equation method combined with the conformable space-time fractional derivatives⁴⁰ for finding many new Jacobi elliptic function solutions, solitons and other solutions of the following nonlinear conformable space-time fractional fourth-order Pochhammer-Chree equation:

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}}{\partial t^{2\alpha}} \left(\frac{\partial^{2\beta} u}{\partial x^{2\beta}} \right) - \frac{\partial^{2\beta}}{\partial x^{2\beta}} \left(\alpha_1 u + \beta_1 u^{n+1} + \gamma_1 u^{2n+1} \right) = 0, \quad n \geq 1, \quad (1.1)$$

Where $0 < \alpha, \beta \leq 1$ and $u(x, t)$ is a real function, while α_1, β_1 and γ_1 are arbitrary constants. Equation (1.1) represents a nonlinear model of longitudinal wave propagation of elastic rods.^{45,46} Here the exponent $n \geq 1$ is the power law nonlinearity parameter. When $\alpha = \beta = 1$, Equation (1.1) has been discussed in⁴² using the generalized projective Riccati equation method, in⁴⁷ using the extended (G'/G) -expansion method, in⁴⁸ using the (G'/G) -expansion method, in⁴⁹ using the tanh-coth and the sine-cosine methods, and in⁵⁰ using the exp-function method.

This article is organized as follows: In Section 2, the description of conformable fractional derivative is given. In Section 3, the description of the unified sub-equation method combined with the conformable space-time fractional derivatives is obtained. In Section 4, we apply this method to the conformable space-time fractional fourth-order Pochhammer-Chree equation (1.1). In Section 5, we present the graphical representations for some solutions of Equation (1.1). In Section 6, conclusions are obtained. To the best of our knowledge, Equation (1.1) has not been previously considered in literature using the method of Section 3.

Description of the conformable fractional derivative

Khalil et al.⁵¹ introduced a novel definition of fractional derivative named the conformable fractional derivative, which can rectify the deficiencies of the other definitions.

Definition1: Suppose $f: [0, \infty) \rightarrow R$ is a function. Then, the conformable fractional derivative of f of order α is defined as

$$T_{\alpha}(f)(t) = \lim_{\tau \rightarrow 0} \frac{f(t + \tau t^{1-\alpha}) - f(t)}{\tau}, \quad (2.1)$$

For all $t > 0$ and $\alpha \in (0, 1]$. Several properties of the conformable fractional derivative are given below as in⁵¹⁻⁵³

Theorem 1: Suppose $\alpha \in (0, 1]$, and f and g are α – differentiable at $t > 0$. Then

$$T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g), \quad \forall a, b \in R. \quad (2.2)$$

$$T_{\alpha}(t^{\mu}) = \mu t^{\mu-\alpha}, \quad \forall \mu \in R. \quad (2.3)$$

$$T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f), \quad (2.4)$$

$$T_{\alpha}\left(\frac{f}{g}\right) = \left(\frac{gT_{\alpha}(f) - fT_{\alpha}(g)}{g^2}\right), \quad (2.5)$$

Furthermore, if f is differentiable; then

$$T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \quad (2.6)$$

Theorem 2: Suppose $f: [0, \infty) \rightarrow R$ is a differentiable function and also α – differentiable. Let g be a function defined in the range of f and also differentiable. Then

$$T_{\alpha}(f \circ g)(t) = t^{1-\alpha} g'(t) f'(g(t)). \quad (2.7)$$

Description of the unified sub-equation method combined with the conformable space-time fractional derivatives

Consider the following nonlinear PDE:

$$F \frac{\partial^{\alpha} u}{\partial t^{\alpha}} \frac{\partial^{\beta} u}{\partial x^{\beta}} \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} \frac{\partial^{2\beta} u}{\partial x^{2\beta}} = 0, \quad \alpha, \beta \leq 1 \quad (3.1)$$

Where F is a polynomial in $u(x, t)$ and its partial derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1: We use the conformable space-time wave transformation:

$$u(x, t) = u(\xi), \quad \xi = \frac{x^{\beta}}{\beta} - c_1 \frac{t^{\alpha}}{\alpha}, \quad (3.2)$$

Where c_1 is a constant, to reduce Equation (3.1) to the following ODE:

$$P(u, u', u'', \dots) = 0, \quad (3.3)$$

Where P is a polynomial in $u(\xi)$ and its total derivatives, such that

$$' = \frac{d}{d\xi}.$$

Step 2: We assume that Equation (3.3) has the formal solution:

$$u(\xi) = \sum_{i=1}^N a_i (\xi)^{i-1} \left[\phi(\xi) + \psi(\xi) \right], \quad (3.4)$$

Where a_0, a_i, b_i ($i = 1, \dots, N$) are constants to be determined later, such that $a_N \neq 0$ or $b_N \neq 0$, while $f(\xi)$ and $g(\xi)$ satisfy the auxiliary ODEs:

$$f'(\xi) = f(\xi)g(\xi), \quad (3.5)$$

$$g'(\xi) = q + g^2(\xi) + rf^{-2}(\xi), \quad (3.6)$$

$$g^2(\xi) \frac{d^2 \xi}{d\xi^2} - \left[q + \frac{r}{2} f^{-2}(\xi) + cf^2(\xi) \right] = 0 \quad (3.7)$$

Where q, r and c are constants.

Step 3: We determine the positive integer N in (3.4) by using the homogeneous balance between the highest order derivatives and the nonlinear terms in Equation (3.3). More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = N$, which gives rise to the degree of other expressions as follows:

$$D \left[\left(\frac{d^{q_1} u(\xi)}{d\xi^{q_1}} \right)^{s_1} \right] = s_1 + (q_1 - 1)N. \quad (3.8)$$

From (3.8) we can get the value of N in (3.4). In some nonlinear equations, the balance number N is not a positive integer. In this case, we make the following transformations:

$$\text{When } N = \frac{q_1}{p_1}, \text{ where } \frac{q_1}{p_1} \text{ is a fraction in the lowest terms, we let}$$

$$u(\xi) = [v(\xi)]^{\frac{q_1}{p_1}}, \quad (3.9)$$

When N is a negative number, we let

$$u(\xi) = [v(\xi)]^N, \quad (3.10)$$

And substitute (3.9) or (3.10) into Equation (3.3) to get a new equation in terms of the function $v(\xi)$ with a positive integer balance number.

Step 4: We substitute (3.4) along with (3.5)-(3.7) into Equation (3.3) and collect all terms of the same order of $f^i(\xi) g^j(\xi)$ ($i, j = 0, 1, \dots$) and set them to zero, yield a set of algebraic equations which can be solved by using the Maple or Mathematical to find $a_0, a_i, b_i, c_1, q, r, c$.

Step 5: It is well-known⁴⁰ that (3.5), (3.6) have the following Jacobi elliptic function solutions:

(1) If $q = (1 + m^2)$, $r = -2m^2$, $c = -1$, then

$$f_1(\xi) = \frac{\text{sn}(\xi, m)}{\text{sn}(\xi, m)}, \quad g_1(\xi) = -\frac{\text{cn}(\xi, m) \text{dn}(\xi, m)}{\text{sn}(\xi, m)}.$$

(2) If $q = (1 - 2m^2)$, $r = 2m^2$, $c = (m^2 - 1)$, then

$$f_2(\xi) = \frac{\text{sn}(\xi, m)}{\text{cn}(\xi, m)}, \quad g_2(\xi) = \frac{\text{sn}(\xi, m) \text{dn}(\xi, m)}{\text{cn}(\xi, m)}.$$

(3) If $q = (-2 + m^2)$, $r = 2$, $c = (1 - m^2)$, then

$$f_3(\xi) = \frac{\text{sn}(\xi, m)}{\text{dn}(\xi, m)}, \quad g_3(\xi) = \frac{\text{sn}(\xi, m) \text{dn}(\xi, m)}{\text{dn}(\xi, m)}.$$

(4) If $q = (1 + m^2)$, $r = -2$, $c = -m^2$, then

$$f_4(\xi) = \text{sn}(\xi, m), \quad g_4(\xi) = \frac{\text{cn}(\xi, m) \text{dn}(\xi, m)}{\text{sn}(\xi, m)}.$$

(5) If $q = (1 - 2m^2)$, $r = (-2 + 2m^2)$, $c = m^2$, then

$$f_5(\xi) = \text{cn}(\xi, m), \quad g_5(\xi) = -\frac{\text{sn}(\xi, m) \text{dn}(\xi, m)}{\text{cn}(\xi, m)}.$$

(6) If $q = (-2 + m^2)$, $r = (2 - 2m^2)$, $c = 1$, then

$$f_6(\xi) = \text{dn}(\xi, m), \quad g_6(\xi) = -\frac{\text{sn}(\xi, m) \text{cn}(\xi, m)}{\text{dn}(\xi, m)}.$$

(7) If $q = (-2 + m^2)$, $r = (-2 + 2m^2)$, $c = -1$, then

$$f_7(\xi) = \frac{\text{cn}(\xi, m)}{\text{sn}(\xi, m)}, \quad g_7(\xi) = -\frac{\text{dn}(\xi, m)}{\text{sn}(\xi, m) \text{cn}(\xi, m)}.$$

(8) If $q = (1 - 2m^2)$, $r = (2m^2 - 2m^4)$, $c = -1$, then

$$f_8(\xi) = \frac{\text{dn}(\xi, m)}{\text{sn}(\xi, m)}, \quad g_8(\xi) = -\frac{\text{cn}(\xi, m)}{\text{sn}(\xi, m) \text{dn}(\xi, m)}.$$

(9) If $q = (-2 + m^2)$, $r = -2$, $c = (-1 + m^2)$, then

$$f_9(\xi) = \frac{\text{sn}(\xi, m)}{\text{cn}(\xi, m)}, \quad g_9(\xi) = \frac{\text{dn}(\xi, m)}{\text{sn}(\xi, m) \text{cn}(\xi, m)}.$$

(10) If $q = (1 + m^2)$, $r = -2m^2$, $c = -1$, then

$$f_{10}(\xi) = \frac{\text{dn}(\xi, m)}{\text{cn}(\xi, m)}, \quad g_{10}(\xi) = \frac{(1 - m^2) \text{sn}(\xi, m)}{\text{cn}(\xi, m) \text{dn}(\xi, m)}.$$

(11) If $q = (1 - 2m^2)$, $r = -2$, $c = (m^2 - m^4)$, then

$$f_{11}(\xi) = \frac{\text{sn}(\xi, m)}{\text{dn}(\xi, m)}, \quad g_{11}(\xi) = \frac{\text{cn}(\xi, m)}{\text{sn}(\xi, m) \text{dn}(\xi, m)}.$$

(12) If $q = \frac{1}{2}(-1 + 2m^2)$, $r = -\frac{1}{2}$, $c = -\frac{1}{4}$, then

$$f_{12}(\xi) = \frac{\text{cn}(\xi, m) \pm 1}{\text{sn}(\xi, m)}, \quad g_{12}(\xi) = \mp \text{ds}(\xi, m).$$

(13) If $q = -\frac{1}{2}(m^2 + 1)$, $r = \frac{1}{2}(1 - m^2)$, $c = \frac{1}{4}(1 - m^2)$, then

$$f_{13}(\xi) = \frac{\text{dn}(\xi, m)}{\text{msn}(\xi, m) \pm 1}, \quad g_{13}(\xi) = \mp \text{mcd}(\xi, m).$$

(14) If $q = -\frac{1}{2}(1 + m^2)$, $r = \frac{1}{2}(m^2 - 1)^2$, $c = \frac{1}{4}$, then

$$f_{14}(\xi) = \text{mcn}(\xi, m) + \text{dn}(\xi, m), \quad g_{14}(\xi) = -\text{msn}(\xi, m).$$

(15) If $q = -\frac{1}{2}(1 + m^2)$, $r = -\frac{1}{2}(m^2 - 1)^2$, $c = -\frac{1}{4}$, then

$$f_{15}(\xi) = \frac{\text{cn}(\xi, m) \pm \text{dn}(\xi, m)}{\text{sn}(\xi, m)}, \quad g_{15}(\xi) = \mp \text{ns}(\xi, m).$$

(16) If $q = (m^2 - 6m + 1)$, $r = -2$, $c = 4m(m - 1)^2$, then

$$f_{16}(\xi) = \frac{\text{sn}(\xi, m)}{\text{msn}^2(\xi, m) - 1}, \quad g_{16}(\xi) = -\text{cs}(\xi, m) \text{dn}(\xi, m) \left[\frac{\xi}{\text{msn}^2(\xi, m) - 1} + \right].$$

(17) If $q = (m^2 + 6m + 1)$, $r = -2$, $c = -4m(m + 1)^2$, then

$$f_{17}(\xi) = \frac{\text{sn}(\xi, m)}{\text{msn}^2(\xi, m) + 1}, \quad g_{17}(\xi) = -\text{cs}(\xi, m) \text{dn}(\xi, m) \left[\frac{\xi}{\text{msn}^2(\xi, m) + 1} - \right].$$

(18) If $q = -\frac{1}{2}(1 + m^2)$, $r = \frac{1}{2}(m^2 - 1)$, $c = \frac{1}{4}(m^2 - 1)$, then

$$f_{18}(\xi) = \frac{\text{cn}(\xi, m)}{\text{sn}(\xi, m) \pm 1}, \quad g_{18}(\xi) = \mp \text{dc}(\xi, m).$$

(19) If $q = \frac{1}{2}(2 - m^2)$, $r = -\frac{m^4}{2}$, $c = -\frac{1}{4}$, then

$$f_{19}(\xi) = \frac{dn(\xi, m) \pm 1}{sn(\xi, m)}, \quad g_{19}(\xi) = \mp cs(\xi, m).$$

(20) If $q = \frac{1}{2}(2m^2 - 1)$, $r = -\frac{1}{2}$, $c = -\frac{1}{4}$, then

$$f_{20}(\xi) = \frac{sn(\xi, m)}{1 \pm cn(\xi, m)}, \quad g_{20}(\xi) = \pm ds(\xi, m).$$

(21) If $q = -\frac{1}{2}(1 + m^2)$, $r = -\frac{1}{2}$, $c = -\frac{1}{4}(m^2 - 1)^2$, then

$$f_{21}(\xi) = \frac{sn(\xi, m)}{cn(\xi, m) \pm dn(\xi, m)}, \quad g_{21}(\xi) = \pm ns(\xi, m).$$

Where $sn(\xi, m)$, $cn(\xi, m)$ and $dn(\xi, m)$ are Jacobi elliptic sine function, Jacobi elliptic cosine function, Jacobi elliptic function of the third kind respectively, and m denotes the modulus of Jacobi elliptic

functions, where $0 \leq m \leq 1$. It is well-known⁵⁴⁻⁵⁶ that the Jacobi-elliptic functions satisfy the following relations:

$$\begin{aligned} cn^2(\xi, m) &= 1 - sn^2(\xi, m), \quad dn^2(\xi, m) = 1 - m^2 sn^2(\xi, m), \\ sn'(\xi, m) &= cn(\xi, m)dn(\xi, m), \quad cn'(\xi, m) = -sn(\xi, m)dn(\xi, m), \\ dn'(\xi, m) &= -m^2 sn(\xi, m)cn(\xi, m). \end{aligned}$$

$$\begin{aligned} ns(\xi, m) &= \frac{1}{sn(\xi, m)}, \quad nc(\xi, m) = \frac{1}{cn(\xi, m)}, \quad nd(\xi, m) = \frac{1}{dn(\xi, m)}, \\ sc(\xi, m) &= \frac{sn(\xi, m)}{cn(\xi, m)}, \quad sd(\xi, m) = \frac{sn(\xi, m)}{dn(\xi, m)}, \quad cs(\xi, m) = \frac{cn(\xi, m)}{sn(\xi, m)}, \\ cd(\xi, m) &= \frac{cn(\xi, m)}{dn(\xi, m)}, \quad ds(\xi, m) = \frac{dn(\xi, m)}{sn(\xi, m)}, \quad dc(\xi, m) = \frac{dn(\xi, m)}{cn(\xi, m)}. \end{aligned}$$

The Jacobi elliptic functions degenerate into hyperbolic functions when $m \rightarrow 1$ as follows:

$$\begin{aligned} sn(\xi, 1) &\rightarrow \tanh(\xi), \quad cn(\xi, 1) \rightarrow \operatorname{sech}(\xi), \quad dn(\xi, 1) \rightarrow \operatorname{sech}(\xi), \quad ns(\xi, 1) \rightarrow \coth(\xi), \quad dc(\xi, 1) \rightarrow 1, \\ ds(\xi, 1) &\rightarrow \operatorname{cosech}(\xi), \quad sc(\xi, 1) \rightarrow \sinh(\xi), \quad sd(\xi, 1) \rightarrow \sinh(\xi), \quad cs(\xi, 1) \rightarrow \operatorname{cosech}(\xi), \end{aligned}$$

And into trigonometric functions when $m \rightarrow 0$ as follows:

$$\begin{aligned} sn(\xi, 0) &\rightarrow \sin(\xi), \quad cn(\xi, 0) \rightarrow \cos(\xi), \quad dn(\xi, 0) \rightarrow 1, \quad ns(\xi, 0) \rightarrow \operatorname{cosec}(\xi), \quad cs(\xi, 0) \rightarrow \cot(\xi), \\ ds(\xi, 0) &\rightarrow \operatorname{cosec}(\xi), \quad sc(\xi, 0) \rightarrow \tan(\xi), \quad sd(\xi, 0) \rightarrow \sin(\xi), \quad dc(\xi, 0) \rightarrow \sec(\xi) \end{aligned}$$

Step 6: We substitute the values a_0, a, b and the solutions (1)–(21) given in step 5 into (3.4), to get the exact solutions of Equation (3.1).

Applications

In this section, we apply the method of section 3, to solve Equation (1.1). To this aim, we first use the conformable space-time wave transformation:

$$u(x, t) = u(\xi), \quad \xi = \frac{x^\beta}{\beta} - c_1 \frac{t^\alpha}{\alpha}, \quad (4.1)$$

Where c_1 is a non zero constant and $0 < \alpha, \beta \leq 1$, to reduce Equation (1.1) into the following nonlinear ordinary differential equation (ODE):

$$D_1^2 D_1^2 u - \frac{2}{1} u''' - \left(\alpha_1 + \beta_1 u^{n+1} + \gamma_1 u^{2n+1} \right)'' = 0, \quad (4.2)$$

Integrating Equation (4.2) twice with respect to ξ and vanishing the constants of integration, we get

$$(c_1^2 - \alpha_1)u - c_1^2 u'' - \beta_1 u^{n+1} - \gamma_1 u^{2n+1} = 0, \quad (4.3)$$

Balancing u'' with u^{2n+1} , we get $N = \frac{1}{n}$. According to (3.9) we use the transformation:

$$u(\xi) = v^{\frac{1}{n}}(\xi), \quad (4.4)$$

Where $v(\xi)$ is a new function of ξ , to reduce Equation (4.3) into the new ODE:

$$(c_1^2 - \alpha_1)n^2 v^2 - c_1^2 n v v'' - c_1^2 (1 - n) v'^2 - \beta_1 n^2 v^3 - \gamma_1 n^2 v^4 = 0, \quad (4.5)$$

Balancing $v v''$ with v^4 in Equation (4.5), we get $N = 1$. According to the form (3.4), Equation (4.5) has the formal solution:

Equation (4.5) and collecting all terms of the same order of $f^i(\xi)$, $g^j(\xi)$, $(i, j = 0, 1, 2, \dots)$ and setting them to zero, we have the following algebraic equations:

following algebraic equations:

(4.7)

Result 1: If we substitute $q = (1 + m^2)$, $r = -2m^2$, $c = -1$ into the algebraic equations (4.7) and use the Maple, we have

while if $m \rightarrow 1$, then we have the solitary wave solution:

or

Substituting (4.9) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left(1 - \frac{1}{msn(\xi, m)} \right) \right\}^{\frac{1}{2}}, \quad \gamma_1 < 0, \beta_1 > 0 \quad (4.13)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4m\sqrt{\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the singular soliton solution:

Where $\xi = \frac{x^\beta}{\alpha} - \left(\sqrt{\frac{-\alpha_1}{(1+2m^2)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 0$, then we have the periodic solution:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 - \coth \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right] \right\}^{\frac{1}{2}}. \quad (4.14)$$

Result 2: If we substitute $q = (1 - 2m^2)$, $r = 2m^2$, $c = (m^2 - 1)$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ m = m, n = 2, a_0 = \frac{-3\beta_1}{8\gamma_1}, a_1 = \frac{3\beta_1}{8\gamma_1}, c_1 = \frac{\beta_1}{4} \sqrt{\frac{3}{\gamma_1(m^2 - 1)}}, \alpha_1 = \frac{3\beta_1^2(4m^2 - 1)}{64\gamma_1(m^2 - 1)}, b_1 = 0, \gamma_1(m^2 - 1) > 0, \right\} \quad (4.15)$$

Substituting (4.15) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left(-\frac{1}{cn(\xi, m)} \right) \right\}^{\frac{1}{2}} \gamma_1 \beta_1 \quad (4.16)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4} \sqrt{\frac{3}{\gamma_1(m^2 - 1)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 0$, then we

have the periodic solution:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 - \sec \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right] \right\}^{\frac{1}{2}}, \beta_1 \gamma_1 < 0. \quad (4.17)$$

Result 3: If we substitute $q = (-2 + m^2)$, $r = 2$, $c = (1 - m^2)$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \beta_1 = 1, \beta_{\text{BB}} = 0, \gamma_1 = 0, \alpha_1 = \sqrt{\frac{2\alpha_1}{\gamma_1(2m^2 - 5)}}, \alpha = \sqrt{\frac{-\alpha_1}{(2m^2 - 5)}}, \alpha > 0, \gamma < 0, \right\} \quad (4.18)$$

Substituting (4.18) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = m^2 \sqrt{\frac{2\alpha_1}{\gamma_1(2m^2 - 5)}} \left(\frac{sn(\xi, m) cn(\xi, m)}{dn(\xi, m)} \right), \alpha_1 > 0, \gamma_1 < 0 \quad (4.19)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\sqrt{\frac{-\alpha_1}{(2m^2 - 5)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the dark soliton solution:

$$u(x, t) = \sqrt{\frac{-2\alpha_1}{3\gamma_1}} \left[\left(\frac{x^\beta}{\beta} - \sqrt{\frac{\alpha_1}{3}} \frac{t^\alpha}{\alpha} \right) \right] \quad (4.20)$$

Result 4: If we substitute $q = (1 + m^2)$, $r = -2$, $c = -m^2$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ m = m, n = 2, a_0 = \frac{-3\beta_1}{8\gamma_1}, b_1 = 0, a_1 = \frac{-3\beta_1}{8\gamma_1}, \alpha = \left\{ \frac{15\beta_1^2(m^2 - 1)}{64\gamma_1 m^2}, c_1 = \frac{\beta_1}{4m} \sqrt{\frac{-3}{\gamma_1}}, \gamma_1 < 0 \right\} \right\} \quad (4.21)$$

Substituting (4.21) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} (1 + sn(\xi, m)) \right\}^{\frac{1}{2}}, \gamma_1 < 0, \beta_1 > 0 \quad (4.22)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4m} \sqrt{\frac{-3}{\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the dark soliton solution:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \tanh \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right] \right\}^{\frac{1}{2}}. \quad (4.23)$$

Result 5: If we substitute $q = (1 - 2m^2)$, $r = (-2 + 2m^2)$, $c = m^2$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ n = 1, a_0 = \frac{-\beta_1}{3\gamma_1}, a_1 = 0, b_1 = \frac{-\beta_1}{3\gamma_1 \sqrt{2m^2 - 1}}, c_1 = \frac{\beta_1}{3} \sqrt{\frac{-1}{\gamma_1(m^2 - 1)}}, \alpha_1 = \frac{\beta_1^2(8m^2 - 5)}{\gamma_1(m^2 - 1)}, \gamma_1(2m^2 - 1) < 0, \right\} \quad (4.24)$$

Substituting (4.24) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \frac{-\beta_1}{3\gamma_1} \left[1 - \frac{1}{\sqrt{2m^2 - 1}} \left(\frac{\text{sn}(\xi, m) \text{dn}(\xi, m)}{\text{cn}(\xi, m)} \right) \right] \quad (4.25)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{3} \sqrt{\frac{-1}{2\gamma_1(2m^2 - 1)}} \right) \frac{t^\alpha}{\alpha}$, $\gamma_1 > 0$ or $\gamma_1 < 0$. If $m \rightarrow 1$, then we have the dark soliton solution:

$$u(x, t) = \frac{-\beta_1}{3\gamma_1} \left[1 - \tanh \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{3} \sqrt{\frac{-1}{2\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right], \quad \gamma_1 < 0. \quad (4.26)$$

Result 6: If we substitute $q = (-2 + m^2)$, $r = (2 - 2m^2)$, $c = 1$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} \ddot{u}\ddot{u}\ddot{u} &= 2, \alpha_1 = \frac{3\beta_1^2(m^2 + 8)}{64\gamma_1}, \beta_0 = \frac{-3\beta_1}{8\gamma_1}, \beta_1 = \frac{-3\beta_1}{8\gamma_1}, \beta_2 = 0, \beta_3 = \frac{\beta_1}{4}\sqrt{\frac{3}{\gamma_1}}, \gamma_1 > 0 \end{aligned} \right\} \quad (4.27)$$

Substituting (4.27) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} (1 + \text{dn}(\xi, m)) \right\}^{\frac{1}{2}}, \quad \gamma_1 > 0, \beta_1 < 0. \quad (4.28)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4} \sqrt{\frac{3}{\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the bright soliton solution:

$$\ddot{u}\ddot{u} = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{4} \sqrt{\frac{3}{\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right] \right\}^{\frac{1}{2}}. \quad (4.29)$$

Result 7: If we substitute $q = (-2 + m^2)$, $r = (-2 + 2m^2)$, $c = -1$

into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} m &= m, n = 2, b_1 = \frac{3\beta_1}{8m^2\gamma_1} (-1 + \sqrt{1 - m^2}), a_0 = \frac{-3\beta_1}{8\gamma_1}, c_1 = \frac{\beta_1}{4m^2} \sqrt{\frac{3}{\gamma_1} (m^2 - 2 + 2\sqrt{1 - m^2})}, \\ \alpha_1 &= \frac{3\beta_1^2}{16m^4\gamma_1} \left(\frac{(m^2 - \ddot{u}\ddot{u}\sqrt{-m^2})(-m^4 - m^2 + (m^2 -)\sqrt{-m^2})}{(-1 + \sqrt{1 - m^2})^2} \right), a_1 = 0, (m^2 - 2 + 2\sqrt{1 - m^2})\gamma_1 > 0, \end{aligned} \right\} \quad (4.30)$$

Substituting (4.30) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \left(\frac{-1 + \sqrt{1 - m^2}}{m^2} \right) \left(\frac{\text{dn}(\xi, m)}{\text{sn}(\xi, m) \text{cn}(\xi, m)} \right) \right] \right\}^{\frac{1}{2}} \gamma_1 \beta_1 \quad (4.31)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4m^2} \sqrt{\frac{3}{\gamma_1} (m^2 - 2 + 2\sqrt{1 - m^2})} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the same solitary wave solution (4.14).

Result 8: If we substitute $q = (1 - 2m^2)$, $r = (2m^2 - 2m^4)$, $c = -1$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} m &= m, n = 1, \beta_{\ddot{u}\ddot{u}\ddot{u}} = 0, a = 0, b = 0, a = \sqrt{\frac{\alpha_1}{\gamma_1(m^2 - 1)}}, c = \sqrt{\frac{-\alpha_1}{2(m^2 - 1)}}, \alpha(m^2 - 1) < 0, \gamma < 0 \end{aligned} \right\} \quad (4.32)$$

Substituting (4.32) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \sqrt{\frac{\alpha_1}{\gamma_1(m^2 - 1)}} \left(\frac{\text{dn}(\xi, m)}{\text{sn}(\xi, m)} \right), \quad (4.33)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\sqrt{\frac{-\alpha_1}{2(m^2 - 1)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 0$, then we have the periodic solution:

$$u(x, t) = \sqrt{\frac{-\alpha_1}{\gamma_1}} \left[\text{cosec} \left(\frac{x^\beta}{\beta} - \sqrt{\frac{\alpha_1}{2}} \frac{t^\alpha}{\alpha} \right) \right]. \quad (4.34)$$

Result 9: If we substitute $q = (-2 + m^2)$, $r = -2$, $c = (-1 + m^2)$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} \ddot{u}\ddot{u}\ddot{u} &= 1, \beta_{\ddot{u}\ddot{u}\ddot{u}} = 0, \beta_0 = 0, \beta_1 = 0, \beta_2 = \sqrt{\frac{2\alpha_1}{\gamma_1(2m^2 - 5)}}, \beta_3 = \sqrt{\frac{-\alpha_1}{(2m^2 - 5)}}, \alpha > 0, \gamma < 0 \end{aligned} \right\} \quad (4.35)$$

Substituting (4.35) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \sqrt{\frac{2\alpha_1}{\gamma_1(2m^2-5)}} \left(\frac{dn(\xi, m)}{sn(\xi, m)cn(\xi, m)} \right), \quad \alpha_1 > 0, \gamma_1 < 0 \quad (4.36)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\sqrt{\frac{-\alpha_1}{\gamma_1(2m^2-5)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the singular soliton solution.

$$u(x, t) = \sqrt{\frac{-2\alpha_1}{3\gamma_1}} \left[\coth\left(\frac{x^\beta}{\beta} - \sqrt{\frac{\alpha_1}{3}} \frac{t^\alpha}{\alpha}\right) \right], \quad (4.37)$$

while if $m \rightarrow 0$, then we have the periodic solution:

$$u(x, t) = \sqrt{\frac{-2\alpha_1}{5\gamma_1}} \left[\operatorname{cosec}\left(\frac{x^\beta}{\beta} - \sqrt{\frac{\alpha_1}{5}} \frac{t^\alpha}{\alpha}\right) \sec\left(\frac{x^\beta}{\beta} - \sqrt{\frac{\alpha_1}{5}} \frac{t^\alpha}{\alpha}\right) \right]. \quad (4.38)$$

Result 10: If we substitute $q = (1+m^2)$, $r = -2m^2$, $c = -1$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ m = m, n = 1, a_0 = 0, a_1 = \frac{-3\beta_1}{8\gamma_1} \left(m^2 - \frac{1}{4} \right), b_1 = \frac{-3\beta_1}{8\gamma_1} b_1, c_1 = \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \gamma_1 \right\} \quad (4.39)$$

Substituting (4.39) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \frac{dn(\xi, m)}{cn(\xi, m)} \right] \right\}^{\frac{1}{2}}, \quad \beta_1 > 0, \gamma_1 < 0 \quad (4.40)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 0$, then we have the periodic solution:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \sec\left(\frac{x^\beta}{\beta} - \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \frac{t^\alpha}{\alpha}\right) \right] \right\}^{\frac{1}{2}}. \quad (4.41)$$

Result 11: If we substitute $q = (1-2m^2)$, $r = -2$, $c = (m^2 - m^4)$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} & \ddot{u}\ddot{u}\ddot{u} = 1, \beta_1 = 0, a_0 = 0, a_1 = \sqrt{\frac{-2\alpha_1}{\gamma_1(4m^2-1)}}, \\ & c_1 = \sqrt{\frac{\alpha_1}{\gamma_1(4m^2-1)}} (4m^2-1), \alpha_1 > 0, \gamma_1 < 0, \end{aligned} \right\} \quad (4.42)$$

Substituting (4.42) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \sqrt{\frac{-2\alpha_1}{\gamma_1(4m^2-1)}} \left[\frac{cn(\xi, m)}{sn(\xi, m)dn(\xi, m)} \right], \quad \xi = \frac{x^\beta}{\beta} - \left(\sqrt{\frac{\alpha_1}{\gamma_1(4m^2-1)}} \right) \frac{t^\alpha}{\alpha}, \quad (4.43)$$

If $m \rightarrow 0$, then we have the same periodic solution (4.11), while if

$m \rightarrow 1$, then we have the same singular soliton solution (4.37).

Result 12: If we substitute $q = \frac{1}{2}(1+m^2)$, $r = \frac{-1}{2}$, $c = \frac{-1}{4}$ into the algebraic equations (4.7) and use the Maple, we have:

$$\left\{ m = m, n = 1, a_0 = 0, a_1 = \frac{\alpha_1}{\gamma_1(m^2-1)}, c_1 = \sqrt{\frac{-\alpha_1}{2(m^2-1)}}, \alpha_1(m^2-1) < 0, \gamma_1 < 0 \right\} \quad (4.44)$$

Substituting (4.44) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \mp \left[\sqrt{\frac{\alpha_1}{\gamma_1(m^2-1)}} ds(\xi, m) \right], \quad (3.45)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\sqrt{\frac{-\alpha_1}{2(m^2-1)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 0$, then we have the same periodic solution (4.34).

Result 13: If we substitute $q = \frac{-1}{2}(m^2+1)$, $r = \frac{1}{2}(1-m^2)$, $c = \frac{1}{4}(1-m^2)$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ m = m, n = 1, a_0 = 0, b_1 = 0, \beta_1 = 0, a_1 = \sqrt{\frac{\alpha_1}{\gamma_1}}, c_1 = \sqrt{\frac{-2\alpha_1}{\gamma_1(m^2-1)}}, \alpha_1(m^2-1) < 0, \gamma_1 \alpha_1 > 0 \right\} \quad (4.46)$$

Substituting (4.46) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \sqrt{\frac{\alpha_1}{\gamma_1}} \left[\frac{dn(\xi, m)}{msn(\xi, m) \pm 1} \right], \quad (4.47)$$

$$\text{where } \xi = \frac{x^\beta}{\beta} - \left(\sqrt{\frac{-2\alpha_1}{\gamma_1(m^2-1)}} \right) \frac{t^\alpha}{\alpha}.$$

Result 14: If we substitute $q = \frac{-1}{2}(1+m^2)$, $r = \frac{1}{2}(m^2-1)$, $c = \frac{1}{4}$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} & \ddot{u}\ddot{u}\ddot{u}\ddot{u} = \alpha \frac{\beta_1^2(1+2m^2)}{9\gamma_1(m^2+1)} - \frac{\beta_1}{3\gamma_1} \left(\frac{-\beta_1}{3\gamma_1} \sqrt{\frac{2}{(m^2+1)}} \right) \\ & , c_1 = \frac{\beta_1}{3} \sqrt{\frac{-1}{\gamma_1(m^2+1)}}, \gamma_1 < 0 \end{aligned} \right\} \quad (4.48)$$

Substituting (4.48) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \frac{-\beta_1}{3\gamma_1} \left[1 - m \sqrt{\frac{2}{(m^2+1)}} sn(\xi, m) \right], \quad (4.49)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{3} \sqrt{\frac{-1}{\gamma_1(m^2+1)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the

same dark soliton solution (4.26).

Result 15: If we substitute $q = \frac{-1}{2}(1+m^2)$, $r = \frac{-1}{2}(m^2-1)^2$, $c = \frac{-1}{4}$

into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{array}{l} m = m, n = 2, \alpha_1 = \frac{-3\beta_1^2(m^2-1)}{64\gamma_1}, a_0 = \frac{-3\beta_1}{8\gamma_1}, a_1 = 0, \\ b_1 = \frac{-3\beta_1}{8\gamma_1}, c_1 = \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}}, \gamma_1 < 0 \end{array} \right\} \quad (4.50)$$

Substituting (4.50) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 \mp \text{ns}(\xi, m) \right] \right\}^{\frac{1}{2}}, \quad \beta_1 > 0, \gamma_1 < 0. \quad (4.51)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the singular soliton solution:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 \mp \coth \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right] \right\}^{\frac{1}{2}}, \quad (4.52)$$

while if $m \rightarrow 0$, then we have the periodic solutions.

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 \mp \text{cosec} \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right] \right\}^{\frac{1}{2}}. \quad (4.53)$$

Result 16: If we substitute $q = \frac{-1}{2}(m^2-1)$, $r = -2$, $c = 4m(m-1)^2$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{array}{l} m = m, n = 1, \beta_1 = 0, a_0 = 0, a_1 = 0, b_1 = \sqrt{\frac{2\alpha_1}{\gamma_1(2m^2-12m+1)}} \\ c_1 = \sqrt{\frac{-\alpha_1}{(2m^2-12m+1)}}, \alpha_1(2m^2-12m+1) < 0, \gamma_1 < 0 \end{array} \right\} \quad (4.54)$$

Substituting (4.54) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \sqrt{\frac{2\alpha_1}{\gamma_1(2m^2-12m+1)}} \left[\text{cs}(\xi, m) \text{dn}(\xi, m) \left(\frac{\text{msn}^2(\xi, m) + 1}{\text{msn}^2(\xi, m) - 1} \right) \right], \quad (4.55)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\alpha_1}{\sqrt{(2m^2-12m+1)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 0$, then we have

the same periodic solution (4.11), while if $m \rightarrow 1$, then we have the solitary wave solutions:

$$u(x, t) = -\sqrt{\frac{-2\alpha_1}{9\gamma_1}} \left[\coth \left(\frac{x^\beta}{\beta} - \frac{\sqrt{\alpha_1}}{3} \frac{t^\alpha}{\alpha} \right) + \tanh \left(\frac{x^\beta}{\beta} - \frac{\sqrt{\alpha_1}}{3} \frac{t^\alpha}{\alpha} \right) \right]. \quad (4.56)$$

Result 17: If we substitute $q = \frac{-1}{2}(m^2+1)$, $r = -2$, $c = -4m(m+1)^2$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{array}{l} m = m, n = 2, a_0 = \frac{-3\beta_1}{8\gamma_1}, a_1 = \frac{-3\beta_1}{8\gamma_1}(m+1), b_1 = 0, \alpha_1 = \frac{-3\beta_1^2(m^2-18m+5)}{256m\gamma_1} \\ c_1 = \frac{\beta_1}{8} \sqrt{\frac{-3}{m\gamma_1}}, \gamma_1 < 0 \end{array} \right\} \quad (4.57)$$

Substituting (4.57) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \frac{(m+1)\text{sn}(\xi, m)}{\text{msn}^2(\xi, m) + 1} \right] \right\}^{\frac{1}{2}}, \quad \beta_1 > 0, \gamma_1 < 0 \quad (4.58)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{8} \sqrt{\frac{-3}{m\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the solitary wave solution:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \frac{2 \tanh \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{8} \sqrt{\frac{-3}{m\gamma_1}} \frac{t^\alpha}{\alpha} \right)}{1 + \tanh^2 \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{8} \sqrt{\frac{-3}{m\gamma_1}} \frac{t^\alpha}{\alpha} \right)} \right] \right\}^{\frac{1}{2}} \quad (4.59)$$

Result 18: If we substitute $q = \frac{-1}{2}(1+m^2)$, $r = \frac{1}{2}(m^2-1)$, $c = \frac{1}{4}(m^2-1)$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{array}{l} m = m, n = 2, a_0 = \frac{-3\beta_1}{8\gamma_1}, b_1 = \frac{-3\beta_1}{8\gamma_1}, a_1 = 0, \\ \alpha_1 = \frac{-3\beta_1^2(m^2-1)}{64\gamma_1}, c_1 = \frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}}, \gamma_1 < 0 \end{array} \right\} \quad (4.60)$$

Substituting (4.60) into (4.4), (4.6), we have the Jacobi elliptic solutions:

periodic solutions (4.17) and (4.41) respectively.

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 \mp \operatorname{dc}(\xi, m) \right] \right\}^{\frac{1}{2}}, \quad \beta_1 > 0, \gamma_1 < 0 \quad (4.61)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{4} \sqrt{\frac{-3}{\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 0$, then we have the same

Result 19: If we substitute $q = \frac{1}{2}(2 - m^2)$, $r = \frac{-1}{2}m^4$, $c = \frac{-1}{4}$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} & \alpha_1 = \frac{-3\beta_1^2}{16\gamma_1 m^4} \left(\frac{\left((10 - m^2)\sqrt{1 - m^2} + m^4 + 6m^2 - 10 \right) (m^2 - 2 + 2\sqrt{1 - m^2})}{(-1 + \sqrt{1 - m^2})^2} \right), b_1 = 0, (3m^2 + 6\sqrt{1 - m^2} - 6)\gamma_1 > 0, \\ & \text{where } \xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{2m^2} \sqrt{\frac{3m^2 + 6\sqrt{1 - m^2} - 6}{\gamma_1}} \right) \frac{t^\alpha}{\alpha}. \end{aligned} \right. \quad (4.62)$$

Substituting (4.62) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 - \left(\frac{-1 + \sqrt{1 - m^2}}{m^2} \right) \left(\frac{\operatorname{dn}(\xi, m) \pm 1}{\operatorname{sn}(\xi, m)} \right) \right] \right\}^{\frac{1}{2}}, \quad \beta_1 \gamma_1 < 0 \quad (4.63)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{2m^2} \sqrt{\frac{3m^2 + 6\sqrt{1 - m^2} - 6}{\gamma_1}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the solitary wave solutions:

$$u(x, t) = \left\{ \frac{-3\beta_1}{8\gamma_1} \left[1 + \operatorname{cosech} \left(\frac{\beta}{2} - \frac{\beta_1}{2} \sqrt{\frac{-\alpha}{\gamma_1}} \right) \pm \coth \left(\frac{\beta}{2} - \frac{\beta_1}{2} \sqrt{\frac{-\alpha}{\gamma_1}} \right) \right] \right\}^{\frac{1}{2}}. \quad (4.64)$$

Result 20: If we substitute $q = \frac{1}{2}(2m^2 - 1)$, $r = \frac{-1}{2}$, $c = \frac{-1}{4}$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ m = m, n = 1, a_{000} = 0, b = 0, \beta = 0, a = \sqrt{\frac{-\alpha_1}{\gamma_1(2m^2 + 1)}}, c = \sqrt{\frac{2\alpha_1}{(2m^2 + 1)}}, \alpha > 0, \gamma < 0 \right\} \quad (4.65)$$

Substituting (4.65) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \sqrt{\frac{-\alpha_1}{\gamma_1(2m^2 + 1)}} \left(\frac{\operatorname{sn}(\xi, m)}{1 \pm \operatorname{cn}(\xi, m)} \right), \quad \alpha_1 > 0, \gamma_1 < 0 \quad (4.66)$$

$$u(x, t) = \sqrt{\frac{-\alpha_1}{\gamma_1}} \left[\frac{\tan \left(\frac{x^\beta}{\beta} - \sqrt{\frac{2\alpha_1}{\gamma_1}} \frac{t^\alpha}{\alpha} \right)}{\frac{x^\beta}{\beta} - \sqrt{\frac{\alpha_1}{\gamma_1}} \frac{t^\alpha}{\alpha} \pm 1} \right]. \quad (4.68)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\sqrt{\frac{2\alpha_1}{\gamma_1(2m^2 + 1)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the solitary wave solution:

$$u(x, t) = \sqrt{\frac{-\alpha_1}{3\gamma_1}} \left[\frac{\tanh \left(\frac{x^\beta}{\beta} - \sqrt{\frac{2\alpha_1}{3\gamma_1}} \frac{t^\alpha}{\alpha} \right)}{1 \pm \operatorname{sech} \left(\frac{x^\beta}{\beta} - \sqrt{\frac{2\alpha_1}{3\gamma_1}} \frac{t^\alpha}{\alpha} \right)} \right], \quad (4.67)$$

while if $m \rightarrow 0$, then we have the periodic solution:

Result 21: If we substitute $q = -\frac{1}{2}(1 + m^2)$, $r = \frac{-1}{2}$, $c = \frac{-1}{4}(m^2 - 1)^2$ into the algebraic equations (4.7) and use the Maple, we have

$$\left\{ \begin{aligned} & m = m, n = 1, a_0 = \frac{-\beta_1}{3\gamma_1}, a_1 = 0, b_1 = \frac{-\beta_1}{3\gamma_1} \sqrt{\frac{2}{(1 + m^2)}}, c_1 = \frac{\beta_1}{3} \sqrt{\frac{-1}{\gamma_1(1 + m^2)}}, \\ & \alpha_1 = \frac{\beta_1^2(1 + 2m^2)}{9\gamma_1(1 + m^2)}, \gamma_1 < 0, \end{aligned} \right. \quad (4.69)$$

Substituting (4.69) into (4.4), (4.6), we have the Jacobi elliptic solutions:

$$u(x, t) = \frac{-\beta_1}{3\gamma_1} \left[1 \pm \sqrt{\frac{2}{(1+m^2)}} \operatorname{ns}(\xi, m) \right], \quad (4.70)$$

where $\xi = \frac{x^\beta}{\beta} - \left(\frac{\beta_1}{3} \sqrt{\frac{-1}{\gamma_1(1+m^2)}} \right) \frac{t^\alpha}{\alpha}$. If $m \rightarrow 1$, then we have the singular soliton solution:

$$u(x, t) = \frac{-\beta_1}{3\gamma_1} \left[1 \pm \coth \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{3} \sqrt{\frac{-1}{2\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right], \quad (4.71)$$

while if $m \rightarrow 0$, then we have the periodic solution:

$$u(x, t) = \frac{-\beta_1}{3\gamma_1} \left[1 \pm \sqrt{2} \operatorname{cosec} \left(\frac{x^\beta}{\beta} - \frac{\beta_1}{3} \sqrt{\frac{-1}{\gamma_1}} \frac{t^\alpha}{\alpha} \right) \right]. \quad (4.72)$$

The graphical representations of some solutions

In this section, we present some graphs of the solitons and other solutions of Eq.(1.1). Let us now examine Figures 1–12 as it illustrates some of our solutions obtained in this article. To this aim, we select some special values of the parameters obtained for example, in some of the solutions of (4.10), (4.12), (4.22), (4.23), (4.28), (4.37), (4.41), (4.56), (4.58), (4.64), (4.67) and (4.72) of the conformable space-time fractional fourth-order Pochhammer-Chree equation (1.1). For more convenience the graphical representations of these solutions are shown in the following figures.

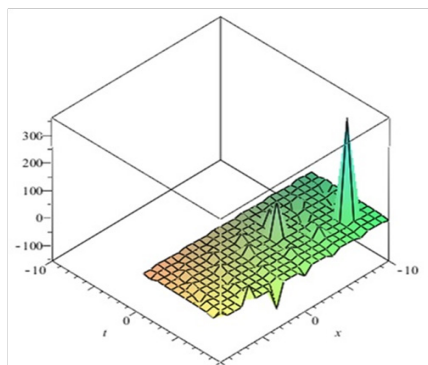


Figure 1 Plot of the Jacobi elliptic conformable fractional solution (4.10) with $\alpha_1 = 2, \alpha = 1, \beta = 1/4, m = 1/2$.

From the above Figures, one can see that the obtained solutions possess the Jacobi elliptic solutions, the conformable fractional solitary wave solution, the Jacobi elliptic conformable fractional solution and the solitary wave solutions. Also, these Figures expressing the behaviour of these solutions which give some perspective readers how the behaviour solutions are produced.

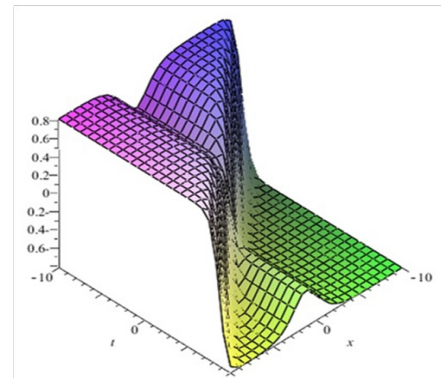


Figure 2 Plot of the solitary wave solution (4.12) with $\alpha_1 = -3, \gamma_1 = -2, \alpha = \beta = 1$.

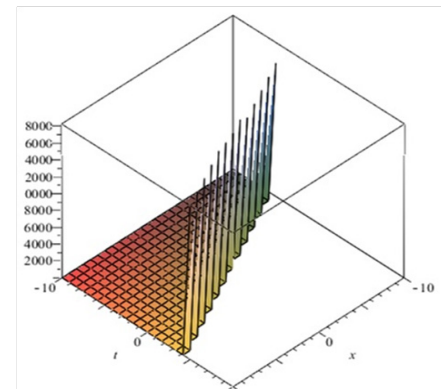


Figure 3 Plot of the Jacobi elliptic conformable fractional solution (4.22) with $\beta_1 = 8, \beta = -3, \alpha = 1, \beta = 1/2, m = 1/3$.

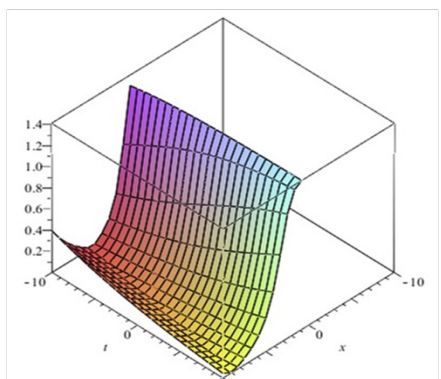


Figure 4 Plot of the solitary wave solution (4.23) with $\beta_1 = 2, \gamma_1 = -3, \alpha = \beta = 1$.

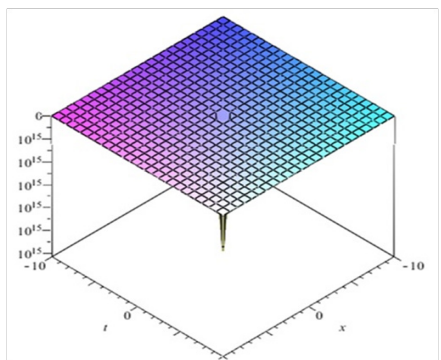


Figure 5 Plot of the Jacobi elliptic conformable fractional solution (4.28) with $\beta_1 = -3, \gamma_1 = 2, \alpha = \beta = 1/2, m = 1/\sqrt{3}$.

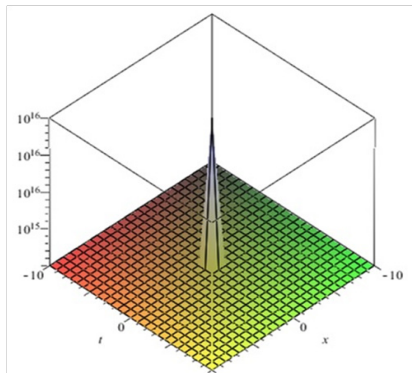


Figure 6 Plot of the solitary wave solution (4.37) with $\alpha_1 = 2, \gamma_1 = -2, \alpha = \beta = 1$.

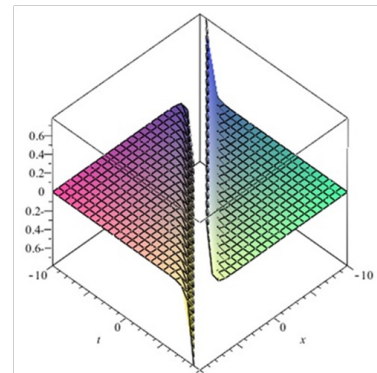


Figure 10 Plot of the solitary wave solution (4.64) with $\beta_1 = 4, \gamma_1 = -3, \alpha = \beta = 1$.

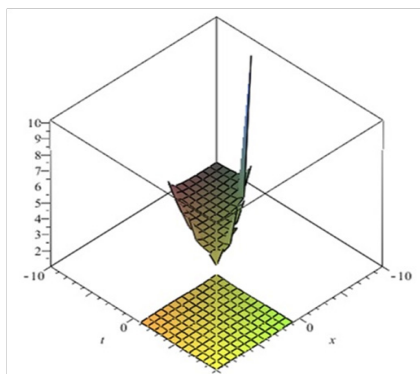


Figure 7 Plot of the periodic wave solution (4.41) with $\beta_1 = 2, \gamma_1 = -3, \alpha = \beta = 1$.

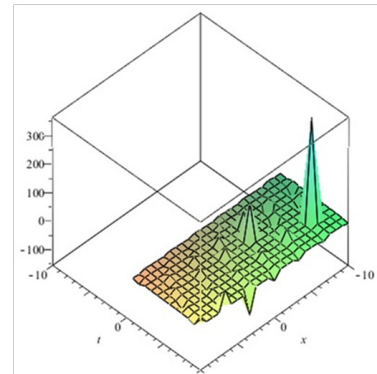


Figure 11 Plot of the solitary wave solution (4.67) with $\alpha_1 = 2, \gamma_1 = -1, \alpha = \beta = 1$.

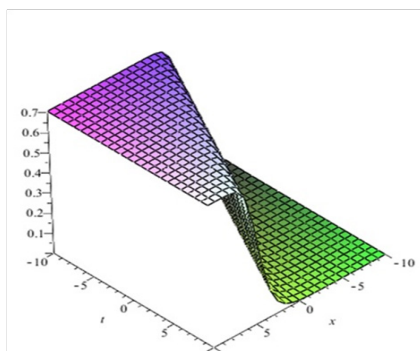


Figure 8 Plot of the solitary wave solution (4.56) with $\alpha_1 = 3, \gamma_1 = -2, \alpha = \beta = 1$.

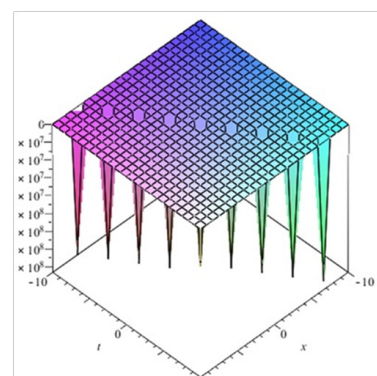


Figure 12 Plot of the periodic wave solution (4.72) with $\beta_1 = 2, \gamma_1 = -1, \alpha = \beta = 1$.

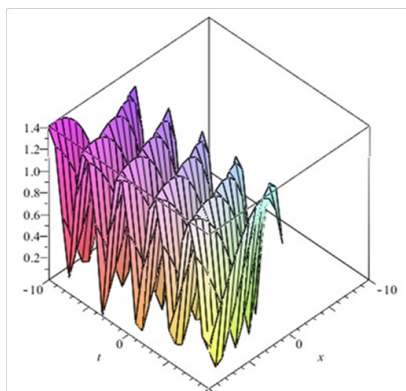


Figure 9 Plot of the Jacobi elliptic conformable fractional solution (4.58) with $\beta_1 = 8, \gamma_1 = -3, \alpha = 1, \beta = \frac{1}{3}, m = \frac{3}{4}$.

Conclusion

We have derived many Jacobi elliptic function solutions, the solitary wave solutions, singular solitary wave solutions and the trigonometric function solutions of the conformable space-time fractional fourth-order Pochhammer-Chree equation (1.1) using the unified sub-equation method combined with the conformable space-time fractional derivatives described in Sec. 3. On comparing our results in this article with that obtained in⁴²⁻⁵⁰ using different methods, we conclude that the Jacobi elliptic solutions obtained in our article are new, while some solitary wave solutions, singular solitary wave solutions and the trigonometric function solutions obtained in our article are equivalent to that obtained in⁴²⁻⁵⁰. From these discussions, we conclude that the proposed method of Sec.3, is direct, concise and effective powerful mathematical tools for obtaining the exact

solutions of other nonlinear evolution equations. Finally, our results in this article have been checked using the Maple by putting them back into the original equation (1.1).⁵⁷

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Conflict of interest

The author declares that there is no conflict of interest.

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