Katugampola-type fractional differential equations with delay and impulses

Abstract

Our aim in this note is to study the existence of solutions of a Katugampola-type fractional impulsive differential equation with delay. We use successive approximation method to show the existence of solutions. In the end, an example is given to verify the hypothetical results.

Keywords: katugampola fractional derivative, impulsive equations, time delay

Introduction

Because of its wide applicability in biology, medicine and in more and more fields, the theory of fractional differential equations (FDEs) has recently been attracting increasing interest, see for instance and references therein. Impulsive differential equations have played an important role in modelling phenomena, especially in describing dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, some authors have used impulse differential systems to describe the model since the last century. For the basic theory on impulsive differential equations, the reader can refer to the books and the papers. In addition, some modelling is done via impulsive functional differential equations when these processes involve hereditary phenomena such as biological and social macro systems. For fractional functional differential equations, the initial value problem, for a class of nonlinear fractional functional differential equations is discussed. For more details, see. Motivated by the papers, the aim of this note is to discuss the existence and uniqueness of solutions of Katugampola-type FDEs with delay and impulses.

Consider the Katugampola-type FDEs with delay and impulse of the form,

\[
\begin{align*}
\mathcal{D}^\alpha \mathcal{Z}(t) &= \mathcal{A}(\mathcal{Z}(t)) + \mathcal{J}(t), \quad t \neq t_k; t \in \mathbb{C} = [0,T]; \\
\mathcal{A}(t_k) &= \mathcal{I}_{t_k}^\alpha \mathcal{Z}(t_k), \quad k = 1,2, ..., m; \\
\mathcal{Z}(t) &= \mathcal{V}(t), \quad t = [-\mu,0],
\end{align*}
\]

where \( \mathcal{D}^\alpha \mathcal{Z} \) is the generalized fractional derivative in Caputo sense, \( \omega \in \mathbb{R} \), \( \beta > 0 \), \( 0 = t_0 < t_1 < ... < t_m < t_{m+1} = T \), \( \mathcal{A} \in \mathbb{C}(\mathbb{R},\mathbb{R}) \) and \( \mathcal{I} \in \mathbb{C}(\mathbb{R},\mathbb{R}) \) are given functions satisfying some assumptions that will be specified later. \( \mathcal{A}(t_k) = \mathcal{I}_{t_k}^\alpha \mathcal{Z}(t_k) \) represents the right and left limits of \( \mathcal{Z}(t) \) at \( t = t_k \), respectively, and they satisfy that \( \mathcal{A}(t_k) = \mathcal{I}_{t_k}^\alpha \mathcal{Z}(t_k) \). If \( \mathcal{Z}(t) \in \mathbb{C}([-\mu,0],\mathbb{R}) \), then for any \( t \in (0,T] \), define \( \mathcal{Z}_2(t) = \mathcal{Z}(t+\theta) \) for \( \theta \in [-\mu,0] \), here \( \mathcal{Z}_2 \) represents the history of the state from time \( t-\mu \) to present time \( t \), \( \mathcal{V} \in \mathbb{C}([-\mu,0],\mathbb{R}) \) and \( \mathcal{V}(0) = 0 \).

The rest of this paper is organized as follows. In Section 2, we give some notations and recall some concepts and preliminary results. In Section 3, the existence and uniqueness of the problem (1) are obtained by successive approximation method. In Section 4, an example is given to demonstrate the effectiveness of the main results.

Preliminaries

In this section, we recollect several definitions of fractional derivatives and integrals from the papers and the papers.

Definition 4.1 The fractional (arbitrary) order integral of the function \( \mathcal{S} \in L^1([a, b], \mathbb{R}^+) \) of order \( \omega \in \mathbb{R}^+ \) is defined by

\[
\mathcal{I}^\omega \mathcal{S}(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1} \mathcal{S}(s) ds,
\]

where \( \Gamma \) is the gamma function.

Definition 4.2 For a function \( \mathcal{S} \) given on the interval \( [a, b] \), the Caputo fractional order derivative of \( \mathcal{S} \), is defined by

\[
\mathcal{D}^\omega \mathcal{S}(t) = \frac{1}{\Gamma(n-\omega)} \int_a^t (t-s)^{n-\omega-1} \mathcal{S}^{(n)}(s) ds,
\]

where \( n = [\omega]+1 \).

Definition 4.3 The generalized left-sided fractional integral \( \mathcal{I}^\alpha \mathcal{S} \)

\[
\mathcal{I}^\alpha \mathcal{S}(t) = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha+1)} \int_t^a (s-t)^{\alpha-1} \mathcal{S}(s) ds,
\]

for \( t > a \), if the integral exists.

Definition 4.4 The generalized fractional derivative, corresponding to the generalized fractional integral (4), is defined for \( 0 < a < t \), by
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\( (D_{a+}^\alpha) S(t) = \frac{1}{\Gamma(n-1)} \int_{a}^{t} (t^\beta - s^\beta)^{n-1} s^{\beta-1} S(s) ds, \) if the integral exists.

**Definition 4.5** The Caputo-type generalized fractional derivative, \( \rho D_{a+}^\alpha S(t) \) is defined via the above generalized fractional derivative (5) as follows

\[
(\rho D_{a+}^\alpha) S(t) = \left( \int_{a}^{t} \frac{(t^\beta - s^\beta)^{n-1} s^{\beta-1} S(s) ds}{(t-a)^{\rho-\alpha}} \right) (t), \quad \text{where } n = \text{Re}(\alpha). \]

**Definition 4.6** The generalized fractional derivative in Caputo sense, corresponding to the generalized fractional integral in Caputo sense (6), is defined for \( 0 < a < t \), by

\[
(\rho D_{a+}^\alpha) S(t) = \left( \int_{a}^{t} \frac{(t^\beta - s^\beta)^{n-1} s^{\beta-1} S(s) ds}{(t^\beta - s^\beta)^{\rho-\alpha}} \right) (t),
\]

\[
= \frac{1}{\Gamma(\rho - \alpha/n)} \left( t^\beta - s^\beta \right)^{\rho-\alpha/n} \int_{a}^{t} \frac{(t^\beta - s^\beta)^{n-1} s^{\beta-1} S(s) ds}{(t^\beta - s^\beta)^{\rho-\alpha}} (t).
\]

**Remark 4.7** In Caputo sense, the Katugampola fractional derivative \( \rho D_{a+}^\alpha \) is a left inverse of the integral operator \( \rho I_{a+}^\alpha \) but in general is not a right inverse.

\[
(\rho D_{a+}^\alpha) (\rho I_{a+}^\alpha) S(t) = S(t)
\]
and the following holds

\[
(\rho D_{a+}^\alpha) (\rho I_{a+}^\alpha) S(t) = S(t) - \sum_{i=0}^{\infty} \frac{(t^\beta - a^\beta)^{i}}{i!} S^{(i)}(a), \quad \text{for } [a, b].
\]

For readers' understanding, we introduce the following notations for the following lemma and theorem.

Let \( \mathcal{J}_{\alpha} = \{0,T\}, \mathcal{J}_{\alpha} = \{0,T\}, \mathcal{J}_{\alpha} = \{t_i, t_{i+1}\}, i = 1, 2, 3, \ldots, m-1, \mathcal{J}_{\alpha} = \{t_{i-1}, t_i\} \) and \( \mathcal{J}_{\alpha} = \{0,T\} \). Obviously, \( \mathcal{P}(\mathcal{J}) \) is a Banach space with the norm \( \| \varphi \|_{\mathcal{P}(\mathcal{J})} = \max_{t \in \mathcal{J}} | \varphi(t) |. \)

**Lemma 4.8** Assume that \( \mathcal{P}(\mathcal{J}) \) is a solution of the initial value problem

\[
\rho D_{a+}^\alpha \varphi(t) = \psi(t), \quad \varphi(t), \varphi'(t) \text{ and } \varphi''(t) \text{ are continuous on } [a, b]; \quad \Delta \mathcal{P}(t_k) = \mathcal{P}(t_k), \quad k = 1, 2, \ldots, \infty
\]

if and only if \( \mathcal{P} \) satisfies the following integral equation

\[
\mathcal{P}(t) = \left[ \rho D_{a+}^\alpha \varphi(t) \right]_{t=a}^{t=T} + \sum_{j=1}^{k} \mathcal{I}(t_j), \quad \text{for } t \in [a, b].
\]

**Proof.** Assume that \( \mathcal{P} \) satisfies (9). One can see, from Remark 2.7 and \( \psi(0)=0 \), that

\[
\mathcal{P}(t) = \mathcal{P}(t) + \sum_{j=1}^{k} \mathcal{I}(t_j), \quad \text{for } t \in [a, b].
\]

In view of \( \mathcal{P}(t_k) = \mathcal{P}(t_k) + \sum_{j=1}^{k} \mathcal{I}(t_j), \) we get that

\[
\mathcal{P}(t) = \mathcal{P}(t) + \sum_{j=1}^{k} \mathcal{I}(t_j), \quad \text{for } t \in [a, b].
\]

Conversely, if \( \mathcal{P} \) is a solution of (10), one can obtain by a direct computation, that \( \rho D_{a+}^\alpha \mathcal{P}(t) = \psi(t), \) \( t \in [0, T], \) and \( \Delta \mathcal{P}(t_k) = \mathcal{P}(t_k), \quad k = 1, 2, \ldots. \)

\[
\mathcal{P}(t) = \mathcal{P}(t) + \sum_{j=1}^{k} \mathcal{I}(t_j), \quad \text{for } t \in [a, b].
\]

Conversely, if \( \mathcal{P} \) is a solution of (10), one can obtain by a direct computation, that \( \rho D_{a+}^\alpha \mathcal{P}(t) = \psi(t), \) \( t \in [0, T], \) and \( \Delta \mathcal{P}(t_k) = \mathcal{P}(t_k), \quad k = 1, 2, \ldots. \)

\[
\mathcal{P}(t) = \mathcal{P}(t) + \sum_{j=1}^{k} \mathcal{I}(t_j), \quad \text{for } t \in [a, b].
\]

This completes the proof.

**Existence and uniqueness results**

Initially, set \( C_0 = \{ \varphi \in \mathcal{P}(\mathcal{J}), \varphi(0) = 0 \} \). For each \( \varphi \in C_0 \), we denote by \( \mathcal{P}(\varphi) \) the function defined by

\[
\mathcal{P}(\varphi)(t) = \rho D_{a+}^\alpha \varphi(t), \quad t \in [0, T], \quad \mathcal{P}(\varphi)(0) = \rho D_{a+}^\alpha \varphi(0) = 0.
\]

If \( \mathcal{P} \) is a solution of (1), then \( \mathcal{P} \) can be decomposed as
\( t \in [0, T] \), where \( \phi(0) = 0 \) and \( \phi(t) = \psi(t) \) for \( -\mu \leq t \leq T \), which implies that \( \mathcal{J} \phi = \mathcal{J} \phi + \phi \) for \( 0 \leq t \leq T \), where

\[
\phi(t) = 0, \quad 0 \leq t \leq T, \quad \text{and} \quad \phi(t) = \psi(t), \quad -\mu \leq t \leq 0.
\]

Thus we have, \( \mathcal{J} \phi = \mathcal{J} \phi + \phi \) for \( 0 \leq t \leq T \), where

\[
\phi(t) = 0, \quad 0 \leq t \leq T, \quad \text{and} \quad \phi(t) = \psi(t), \quad -\mu \leq t \leq 0.
\]
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so the condition

\[ \sum_{i=0}^{p-1} \frac{\alpha_i T^\alpha_i}{\Gamma(\alpha_i)} t^{\alpha_i-1} \int_0^t \frac{d \tau}{\Gamma(\alpha_i)} \left( t^\alpha \right)^{\alpha_i-1} s_{\mu}(s, \tau, x) \, ds = \sum_{i=1}^{p} L_i \left( t \right) \left[ v_1 \right] \]

According to the conditions \((A_i)\), the uniqueness of the problem (1) follows immediately, which completes the proof.

An illustrative example

Consider the following Katugampola-type fractional impulsive differential equation with delay of the form

\[ \frac{d}{dt} 3(t) = \frac{e^{-t}}{(9+e^r)(1+e^r)} 3(t), \quad t \in [0,1], \quad t > \frac{1}{2}, \quad 0 < \omega < 1; \]

\[ \Delta 3(t) = \frac{1}{2} \left( 1 - \frac{1}{2} \right), \quad \omega \leq \beta \leq 0. \]

Let us take, \( \omega = \frac{1}{2}, \rho = 1, \Gamma(\omega + 1) = \frac{4}{10}, \mu \) is a non-negative constant. \( \beta(\theta) = 3(\omega + \theta) \) for \(-\mu \leq \theta \leq 0\) and \(0 \leq \theta \leq 1\).

Set \( \beta(t) = \frac{e^{-t}}{(9+e^r)(1+e^r)} \), \( I (3) = \frac{3}{3+5} \), (for \( t, r \in [0,1] \\cap [0, +\infty) \)).

Now, we can see that

\[ \left| \beta(t, p_1) - \beta(t, q_1) \right| < \frac{e^{-t}}{(9+e^r)(1+e^r)} |p_1 - q_1| < \frac{e^{-t}}{(9+e^r)} |p_1 - q_1|, \]

where \( \alpha(t) = \frac{e^{-t}}{(9+e^r)} \) and \( \alpha_n(t) = \frac{e^{-t}}{(9+e^r)} t \), so the condition \((A_i)\) is satisfied.

On the other hand, we get that

\[ \left| f(p) - f(q) \right| < \frac{3}{3+5} |p - q|, \quad p, q > 0, \]

which satisfies the condition \((A_i)\) of Theorem 3.1 with \( L = \frac{1}{3} \).

By a direct computation, we obtain that

\[ \sum_{i=1}^{p} \frac{\alpha_i T^\alpha_i}{\Gamma(\alpha_i)} t^{\alpha_i-1} \int_0^t \frac{d \tau}{\Gamma(\alpha_i)} \left( t^\alpha \right)^{\alpha_i-1} s_{\mu}(s, \tau, x) \, ds \]

and

\[ \left| \delta(t) \right| < \frac{e^{-t}}{(9+e^r)(1+e^r)} |t| < \frac{1}{10}, \quad t \in [0,1]. \]

As a result, the equations in (20) satisfy all the hypotheses in Theorem 3.1 which guarantees that (20) has a unique solution.

Conclusion

In this note, the existences of solutions of a Katugampola-type fractional impulsive differential equation with delay were investigated. The successive approximation method was employed to show the existence of solutions. The example reflects the applicability of the proposed method.

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Conflict of interest

Author declares that there is no conflict of interest.

References


