

# The effect of temperature–dependent physical properties and fractional thermoelasticity on nonlocal nanobeams

## Abstract

There are many papers are presented in the field of nanoengineering and nanotechnology. In this work, a new model of heat equation with fractional derivatives and relaxation time is introduced to investigate the thermo elastic vibrations of nonlocal nanobeams. In addition, the thermal conductivity and the modulus of elasticity are taken as a linear function of environmental temperature and the body is due to a ramp–type heating. The basic equations of the considered problem are formulated and solved using Laplace transform along with its numerical inversion. The distributions of the temperature, deflection as well as other variable fields are numerically obtained and illustrated graphically. The effects of the nanoscale, fractional order derivative, ramping-time and relaxation time on the considered variables are concerned and discussed in detail. The introduced model can be reduced to the corresponding classical coupled and Lord and Shulman theories for different values of phase lags and fractional order parameters.

**Keywords:** fractional heat equation, nanobeams, temperature–dependent properties, ramp–type heating, fractional order.

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**Ahmed E Abouelregal**

Department of Mathematics, Mansoura University, Egypt

**Correspondence:** Ahmed E Abouelregal, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, Email [ahabogal@mans.edu.eg](mailto:ahabogal@mans.edu.eg)

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## Introduction

In recent years, many interesting models have been established by using fractional calculus to study the physical manners, particularly in the area of diffusion, viscoelasticity, heat conduction, mechanics of solids and biological systems. The use of fractional system derivatives and integrals leads to the formulation of some physical problems that are more economical and useful than the classical approach.

In the second half of the nineteenth century, Caputo,<sup>1</sup> Caputo and Mainardi<sup>2</sup> found an agreement between experimental and theoretical results when using fractional derivatives to describe viscoelastic materials. Povstenko<sup>3</sup> proposed a fractional time derivative heat conduction equation for a quasi–static uncoupled theory of thermo elasticity. He also investigated in<sup>4</sup> a generalizations of the heat conduction and Fourier law by using space and time fractional derivatives. Jiang & Xu<sup>5</sup> obtained a fractional heat equation with a time fractional derivative in the general orthogonal curvilinear and other coordinate system. Samko et al. in their book<sup>6</sup> provided an excellent historical review of this area under discussion. Formulae establishing relations between the two types of fractional derivatives of Riemann–Leuville and Caputo are discussed in detail in.<sup>7,8</sup> It should be confirmed that if attention is given, the results obtained using the Caputo construction can be modified to the Riemann–Liouville version.

The theory of couple thermo elasticity is extended by Lord & Shulman,<sup>9</sup> Green and Lindsay<sup>10</sup> by including the thermal relaxation time in constitutive relations. Green & Naghdi<sup>11</sup> proposed a new generalized thermo elasticity theory by including the thermal–displacement gradient among the independent constitutive variables. An important feature of this theory, which is not present in other thermo elasticity theories, is that it does not accommodate dissipation of thermal energy.

Recently, a completely new fractional order thermo elasticity model involves one thermal relaxation parameter was introduced by Sherief et al.<sup>12</sup> Also, Ezzat<sup>13</sup> established a model of fractional heat

conduction equation by using the new Taylor series expansion of time–fractional order developed by Jumarie.<sup>14</sup> In addition, authors in<sup>15</sup> presented a two generalized models of heat conduction with fractional order for an isotropic nonhomogeneous thermo elastic solid.

Modern structural elements are often exposed to temperature changes of such magnitude that their physical properties are not fixed even in an approximate sense. The thermal and mechanical properties of materials vary with temperature, so that the temperature dependence of material properties must be taken into consideration in the thermal stress analysis of these elements.<sup>16,17</sup> At high temperatures, material properties such as the thermal conductivity, Poisson's ratio, the modulus of elasticity and the thermal expansion coefficients are no longer keep constant.<sup>18</sup> In numerous of papers,<sup>19–24</sup> fractional calculus is often applied in vibration studies of rods, beams, plates as well as in other fields of mechanics.

Micro–scale mechanical resonators have high sensitivity as well as fast response and are widely used as sensors and modulators. Micro and nano–mechanical resonators have attracted considerable attention recently due to their many important technological applications. Accurate analysis of various effects on the characteristics of resonators, such as resonant frequencies and quality factors, is crucial for designing high–performance components. Many authors have studied the vibration and heat transfer process of beams. Some of them are found in the references.<sup>25–27</sup>

The nonlocal theory of elasticity was used to study applications in nano–mechanics including lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics, surface tension fluids, etc. Of all the nanostructures, the mechanical behaviors of nanotubes and nanobeams have been most widely investigated. The models of the nonlocal beams expected increasing attention in the earlier few years. In 1972, Eringen introduced the theory of nonlocal continuum mechanics,<sup>28</sup> in an effort to deal with the small–scale structure problems. The theories of nonlocal continuum consider the state of stress at a point as a function of the states of strain of all points in the body while the classical continuum mechanics

assumes the state of stress at a certain point uniquely depends on the state of strain at that same point. Solutions from various problems support this theory.<sup>28–35</sup>

The current manuscript is an effort to study a thermo elastic problem of a nanobeam loaded thermally by ramp-type heating. Also, a new fractional heat conduction equation model is introduced and the modulus of elasticity is considered to be a linear function of reference temperature. Laplace transform analytical technique is used to calculate the vibration of displacements and temperature. The variations along the axial direction and the through-the-thickness distributions of all fields are investigated. Some comparisons have been also shown graphically to estimate the effects of the small-scale, fractional order and ramping time parameters on all the studied fields.

## Mathematical model for nonlocal thermo elasticity with fractional heat conduction

According to the nonlocal elasticity theory of Eringen, the nonlocal differential constitutive equations for a homogenous thermoelastic materials is<sup>28–30</sup>

$$(1 - \xi \nabla^2) \sigma_{ij} = \tau_{ij} \quad (1)$$

where  $\sigma_{ij}$  and  $\tau_{ij}$  are the nonlocal and local stress tensors, respectively,  $\nabla^2$  is Laplacian operator and  $\xi$  is the nonlocal parameter. One may see that when the internal characteristic length is neglected, i.e., the particles of a medium are considered to be continuously distributed,  $\xi$  is zero, and Eq. (1) reduces to the constitutive equation of classical local thermo elasticity. The classical thermo elasticity is based on the principles of the classical theory of heat conductivity, specifically on the classical Fourier law, which relates the heat flux vector  $q$  to the temperature gradient as follows:

$$q = -K \nabla \theta \quad (2)$$

where  $K$  is the thermal conductivity of a solid,  $\theta = T - T_0$  is the excess temperature distribution, in which  $T_0$  is the environmental temperature. Equation (4) together with the energy equation yields the heat conduction equation or the parabolic heat conduction equation and is diffusive with the notion of infinite speed of propagation of thermal disturbances.

$$\rho C_E \frac{\partial \theta}{\partial t} + \gamma T_0 \frac{\partial}{\partial t} (\text{div}(\mathbf{u})) = -\text{div}(\mathbf{q}) + Q \quad (3)$$

where  $\rho$  is the density,  $C_E$  is the specific heat,  $\mathbf{u}$  is the displacement vector,  $\gamma = \frac{E \alpha_\tau}{1 - 2\nu} = \alpha_\tau (3\lambda + 2\mu)$ ,  $\lambda, \mu$  being Lamé's constants,

$\alpha_\tau$  being the coefficient of linear thermal expansion,  $E$  is Young modulus,  $\nu$  is Poisson's ratio, and  $Q$  is the intensity of heat source. The Riemann–Liouville fractional integral is introduced as a natural generalization of the convolution type integral.<sup>36,37</sup>

$$I^\alpha f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \quad (0 < \alpha < 1). \quad (4)$$

Where  $\Gamma(\alpha)$  is the Gamma function and  $f(t)$  is a Lebesgue integrable continuous function satisfies

$$\lim_{\alpha \rightarrow 1} \frac{d^\alpha}{dt^\alpha} f(t) = f'(t) \quad (5)$$

Using the Taylor series expansion of time fractional order as in,<sup>13</sup> a new non-Fourier model was constructed by<sup>14</sup> in the form

$$\left(1 + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \mathbf{q} = -K \nabla \theta \quad (6)$$

Taking divergence of both sides of Eq. (6), we get

$$\left(1 + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) (\text{div}(\mathbf{q})) = -K \nabla^2 \theta \quad (7)$$

From Eqs. (3) and (7), we can get the fractional ordered generalized heat conduction equation as

$$K \nabla^2 \theta = \left(1 + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha}\right) \left[ \rho C_E \frac{\partial \theta}{\partial t} + \gamma T_0 \frac{\partial}{\partial t} (\text{div}(\mathbf{u})) - Q \right]. \quad (8)$$

The introduced new model of thermo elasticity using the methodology of fractional calculus with wide range  $0 < \alpha \leq 1$  covering two cases of conductivity,  $0 < \alpha < 1$  corresponds to weak conductivity,  $\alpha = 1$  for normal conductivity. For the Lord and Shulman (LS) theory  $\alpha = 1$  and for the classical coupled theory of thermoelasticity (CTE)  $\alpha = 0$  and  $\tau_0 = 0$ .

Constitutive equations:

$$\tau_{ij} = 2\mu e_{ij} + \lambda e_{ij} - \gamma \theta \delta_{ij} \quad (9)$$

Equation of motion:

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i \quad (10)$$

Equations (1) and (8) describe the nonlocal thermo elasticity theory with fractional order derivative. It can be seen that the corresponding local thermoelasticity with fractional order is recovered by putting  $\xi = 0$  in equation (1). It is known that the values of physical properties, such as the moduli of elasticity and thermal expansion coefficients, cannot be assumed constant over a wide range of high-temperature applications. Consider a thermoelastic body of material having temperature-dependent properties on the form<sup>16,38,39</sup>

$$(E, \gamma, K) = (E_0, \gamma_0, K_0) F(\theta) \quad (11)$$

where  $E_0, \gamma_0$  and  $K_0$  are considered to be constants;  $F(\theta)$  is given in a nondimensional function of temperature. In the case of temperature-independent modulus of elasticity,  $F(\theta) = 1$  and  $(E, \gamma, K) = (E_0, \gamma_0, K_0)$ .

The variation of some or all of the these mechanical and thermal properties with temperature can be approximated by linear, exponential, quadratic laws or any other law appropriately approximating the experimental data. We will consider that<sup>16,38,39</sup>

$$F(\theta) = e^{-\beta \theta} \cong 1 - \beta \theta \quad (12)$$

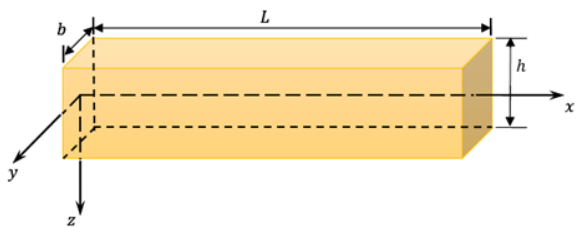
where  $\beta$  is called the empirical material constant measured in  $1/K$ . Equation (12) introduces an empirical model that usually works for a limited range of temperatures.

Since only the infinitesimal temperature deviations from reference temperature are considered and for linearity of the governing partial differential equations of the problem, we have to take into account the condition  $|T - T_0| / T_0 \ll 1$ , which give us the approximating function of  $F(\theta)$  to be in the form.<sup>40,41</sup>

$$F(\theta) \cong F(T_0) = 1 - \beta T_0 \quad (13)$$

## Formulation of the problem and modeling of beam structures

We consider a thermo elastic thin nanobeam initially at temperature  $T_0$  such that  $x$  axis is drawn along the axial direction of the beam and  $y, z$  axes correspond to the width and thickness, respectively (Figure 1). The small flexural deflections of the nanobeam with dimensions of length  $L$ , width  $b$  and thickness  $h$  are considered.



**Figure 1** Schematic diagram for the nanobeam.

The displacement components are given by

$$u = -z \frac{\partial w}{\partial x}, \quad v = 0, \quad w(x, y, z, t) = w(x, t), \quad (14)$$

where  $w$  is the lateral deflection.

For a one-dimensional problem, the differential form of the constitutive Eq. (9) after using Eqs. (1) and (14) can be expressed as:<sup>32,33</sup>

$$\sigma_x - \xi \frac{\partial^2 \sigma_x}{\partial x^2} = -E_0 F(T_0) \left[ \frac{\partial^2 w}{\partial x^2} + \alpha_T \theta \right] \quad (15)$$

where  $\sigma_x$  is the nonlocal axial stress, and  $\alpha_T = \alpha_t / (1 - 2\nu)$ . For transversely vibration of nanobeams, the equilibrium conditions of Euler–Bernoulli theory can be written as

$$\frac{\partial^2 M}{\partial x^2} = \rho A \frac{\partial^2 w}{\partial t^2} \quad (16)$$

where  $A = bh$  is the cross-section area.

With aid of Eq. (15), the flexure moment is given by

$$M(x, t) - \xi \frac{\partial^2 M}{\partial x^2} = -IE_0 F(T_0) \left[ \frac{\partial^2 w}{\partial x^2} + \alpha_T M_T \right] \quad (17)$$

Where  $I = bh^3 / 12$  is the inertia moment of the cross-section,  $IE_0$  is the flexural rigidity of the beam and  $M_T$  is the thermal moment,

$$M_T = \frac{12}{h^3} \int_{-h/2}^{h/2} \theta(x, z, t) z dz \quad (18)$$

Substituting Eq. (17) into Eq. (16), one can get the motion equation of the nanobeam as

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho}{IE_0 F(T_0)} \left( \frac{\partial^2 w}{\partial t^2} - \xi \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) + \alpha_T \frac{\partial^2 M_T}{\partial x^2} = 0 \quad (19)$$

Also, it is exactly seen that the flexure moment of the nonlocal nanobeams is given by

$$M(x, t) = \xi A \rho \frac{\partial^2 w}{\partial t^2} - IE_0 F(T_0) \left[ \frac{\partial^2 w}{\partial x^2} + \alpha_T M_T \right] \quad (20)$$

Substituting the Euler–Bernoulli assumption and Eq. (14) into Eq. (8), gives the generalized heat conduction equation with fractional order derivative without the heat source ( $Q=0$ ), as

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2} = \left( 1 + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left[ \frac{1}{k} \frac{\partial \theta}{\partial t} - \frac{\gamma_0 T_0}{K_0} z \frac{\partial}{\partial t} \left( \frac{\partial^2 w}{\partial x^2} \right) \right] \quad (21)$$

## Solution of the problem

There is no heat flow across the upper and lower surfaces of the nanobeam (thermally insulated), so that  $\frac{\partial \theta}{\partial z}$  should be vanish at the upper and lower surfaces of the nanobeam  $z = \pm h/2$ . For a very nanobeam, assuming that the increment temperature varies in a sinusoidal form along the thickness direction. That is

$$\theta(x, z, t) = \Theta(x, t) \sin\left(\frac{\pi z}{h}\right) \quad (22)$$

Substituting Eq. (22) into Eq. (19), one can get the motion equation of the nanobeams as

$$\frac{\partial^4 w}{\partial x^4} + \frac{\rho A}{E_0 F(T_0)} \left( \frac{\partial^2 w}{\partial t^2} - \xi \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) + \frac{24 \alpha_T}{h \pi^2} \frac{\partial^2 \Theta}{\partial x^2} = 0 \quad (23)$$

Also, the flexure moment can be determined from Eqs. (20) and (22) as

$$M(x, t) = \xi A \rho \frac{\partial^2 w}{\partial t^2} - IE_0 F(T_0) \left[ \frac{\partial^2 w}{\partial x^2} + \frac{24 \alpha_T}{h \pi^2} \Theta \right] \quad (24)$$

Now, multiplying Eq. (21) by means of  $12z/h^3$  and integrating it with respect to  $z$  through the beam thickness from  $-h/2$  to  $h/2$ , yields

$$\frac{\partial^2 \Theta}{\partial x^2} - \frac{\pi^2}{h^2} \frac{\partial^2 \Theta}{\partial z^2} = \left( \delta + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left[ \frac{1}{k} \frac{\partial \Theta}{\partial t} - \frac{\gamma_0 T_0 \pi^2 h}{24 K_0} \frac{\partial}{\partial t} \left( \frac{\partial^2 w}{\partial x^2} \right) \right] \quad (25)$$

Now, for simplicity we will use the following non-dimensional variables:

$$\{x', z', u', w', L', h'\} = c_0 \eta_0 \{x, z, u, w, L, h\}, \quad \{t', \tau_0', \xi'\} = c_0^2 \eta_0 \{t, \tau_0, \eta_0 \xi\},$$

$$\Theta' = \frac{\Theta}{T_0}, \quad \sigma'_x = \frac{\sigma_x}{E_0}, \quad M' = \frac{M}{c_0 \eta_0 IE_0 F(T_0)}, \quad c_0 = \sqrt{\frac{E_0}{\rho}}. \quad (26)$$

So, the basic equations in nondimensional forms are simplified as (dropping the primes for convenience)

$$\frac{\partial^4 w}{\partial x^4} + A_1 \left( \frac{\partial^2 w}{\partial t^2} - \xi \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) = -A_2 \frac{\partial^2 \Theta}{\partial x^2} \quad (27)$$

$$\left( \frac{\partial^2}{\partial x^2} - A_4 \right) \Theta = \left( 1 + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left[ \frac{\partial \Theta}{\partial t} - A_5 \frac{\partial}{\partial t} \left( \frac{\partial^2 w}{\partial x^2} \right) \right]$$

$$M(x, t) = A_1 \left( \xi \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} \right) - A_2 \Theta \quad (28)$$

where

$$A_1 = \frac{12}{F(T_0) h^2}, \quad A_2 = \frac{24 T_0 \alpha_T}{\pi^2 h}, \quad A_3 = \xi A_1, \quad A_4 = \frac{\pi^2}{h^2}, \quad A_5 = \frac{\gamma_0 k \pi^2 h}{24 K_0}$$

## Initial and boundary conditions

To solve the problem, the initial and boundary conditions must be taken into consideration. The homogeneous initial conditions are taken as

$$\Theta(x, 0) = \frac{\partial \Theta(x, 0)}{\partial t} = 0 = w(x, 0) = \frac{\partial w(x, 0)}{\partial t} \quad (29)$$

We will assume that the two ends of the nanobeam are clamped i.e.

$$w(0, t) = w(L, t) = 0 = \frac{\partial^2 w(0, t)}{\partial x^2} = \frac{\partial^2 w(L, t)}{\partial x^2} \quad (30)$$

Also, we consider the nanobeam is loaded thermally by ramp-type heating, which give

$$\Theta(x, t) = \Theta_0 \begin{cases} 0, & t \leq 0, \\ \frac{t}{t_0}, & 0 \leq t \leq t_0, \\ 1, & t > 0 \end{cases} \quad (31)$$

Where  $t_0$  is a non-negative constant called ramp-type parameter and  $\Theta_0$  is a constant. In addition, the temperature at the end boundary should satisfy the following relation

$$\frac{\partial \Theta}{\partial x} = 0 \quad \text{on} \quad x = L \quad (32)$$

## Solution of the problem in the Laplace transform domain

The closed form solution of the governing and constitutive equations can be possible by adapting the Laplace transformation method. Taking the Laplace transform defined by the relation

$$\bar{f}(x, s) = \int_0^\infty f(x, t) e^{-st} dt \quad (33)$$

to both sides of Eqs. (27) and (28) and using the homogeneous initial conditions (29), one gets the field equations in the Laplace transform space as

$$\left( \frac{d^4}{dx^4} - A_3 s^2 \frac{d^2}{dx^2} + A_4 s^2 \right) \bar{w} = -A_2 \frac{d^2 \bar{\Theta}}{dx^2}, \quad (34)$$

$$\left( \frac{\partial^2}{\partial x^2} - B_1 \right) \bar{\Theta} = -B_2 \frac{\partial^2 \bar{w}}{\partial x^2},$$

$$\bar{M}(x, t) = A_1 \left( \xi s^2 \bar{w} - \frac{d^2 \bar{w}}{dx^2} \right) - A_2 \bar{\Theta} \quad (35)$$

where

$$B_1 = s \left( \delta + \frac{s^\alpha \tau_0^\alpha}{\alpha!} \right) + A_4, \quad B_2 = s \left( \delta + \frac{s^\alpha \tau_0^\alpha}{\alpha!} \right) A_5$$

Elimination  $\bar{\Theta}$  or  $\bar{w}$  from Eqs. (34), one obtains:

$$(D^6 - AD^4 + BD^2 - C) \{ \bar{\Theta}, \bar{w} \}(x) = 0, \quad (36)$$

where the coefficients  $A$ ,  $B$  and  $C$  are given by

$$A = A_3 s^2 + A_2 B_2 + B_1, \quad B = A_1 s^2 + B_1 A_3 s^2, \quad C = A_1 B_1 s^2, \quad D = \frac{d}{dx} \quad (37)$$

Equation (36) can be moderated to

$$(D^2 - m_1^2)(D^2 - m_2^2)(D^2 - m_3^2) \{ \bar{\Theta}, \bar{w} \}(x) = 0, \quad (38)$$

where  $m_n^2$ ,  $n = 1, 2, 3, 4$  are roots of

$$m^6 - Am^4 + Bm^2 - C = 0, \quad (39)$$

The solution of the governing equations (39) in the Laplace transformation domain can be represented as

$$\{ \bar{w}, \bar{\Theta} \}(x) = \sum_{n=1}^3 \left( \{ 1, \beta_n \} C_n e^{-m_n x} + \{ 1, \beta_{n+3} \} C_{n+3} e^{m_n x} \right). \quad (40)$$

Where the compatibility between these two equations and Eq. (34), gives

$$\beta_n = -\frac{m_n^2 B_2}{m_n^2 - B_1} \quad (41)$$

where  $C_n$  and  $\beta_n$  are parameters depending on  $s$ .

The axial displacement after using Eq. (40) takes the form

$$\bar{u}(x) = -z \frac{d\bar{w}}{dx} = z \sum_{n=1}^3 m_n \left( C_n e^{-m_n x} - C_{n+3} e^{m_n x} \right). \quad (42)$$

Substituting the expressions of  $\bar{w}$  and  $\bar{\Theta}$  from (40) into (36), we get at the solution for the bending moment  $\bar{M}$  as follows:

$$\bar{M}(x) = \sum_{n=1}^3 \left( A_3 s^2 - m_n^2 - A_2 \beta_n \right) \left( C_n e^{-m_n x} C_{n+3} e^{m_n x} \right). \quad (43)$$

In addition, the strain will be

$$\bar{\epsilon}(x) = \frac{d\bar{u}}{dx} = -z \sum_{n=1}^3 m_n^2 \left( C_n e^{-m_n x} + C_{n+3} e^{m_n x} \right). \quad (44)$$

After using Laplace transform, the boundary conditions (30)–(32) take the forms

$$\begin{aligned} \bar{w}(0, s) &= \bar{w}(L, s) = 0, \\ \frac{\partial^2 \bar{w}(0, s)}{\partial x^2} &= \frac{\partial^2 \bar{w}(L, s)}{\partial x^2} = 0, \\ \frac{\partial \bar{\Theta}(0, s)}{\partial x} &= \Theta_0 \left( \frac{1 - e^{-st_0}}{s^2 t_0} \right) = \bar{G}(s), \\ \frac{\partial \bar{\Theta}(L, s)}{\partial x} &= 0 \end{aligned} \quad (45)$$

Substituting Eq. (40) into the above boundary conditions, one obtains six linear equations;

$$\sum_{n=1}^3 (C_n + C_{n+1}) = 0, \quad (46)$$

$$\sum_{n=1}^3 \left( C_n e^{-m_n L} + C_{n+1} e^{m_n L} \right) = 0, \quad (47)$$

$$\sum_{n=1}^3 m_n^2 (C_n + C_{n+1}) = 0,$$

$$\sum_{n=1}^3 m_n \left( \beta_n C_n - \beta_{n+1} C_{n+1} \right) = -G(s), \quad (48)$$

$$\sum_{n=1}^3 m_n \left( \beta_n C_n e^{-m_n L} - \beta_{n+1} C_{n+1} e^{m_n L} \right) = 0,$$

The solution of the above system of linear equations gives the unknown parameters  $C_n$ , ( $n=1, 2, \dots, 6$ ). In order to determine the studied fields in the physical domain, the Riemann-sum approximation method is used to obtain the numerical results. The details of these methods can be found in Honig & Hirdes.<sup>33</sup>

## Special cases

- The following special cases can be obtained from the system of Eqs. (1), (8) and (9):
- The equations of a coupled theory of nonlocal thermoelasticity with temperature-dependent mechanical properties result from (1), (8), in the limiting case  $\alpha \rightarrow 0$  and (11) by letting  $\tau \rightarrow 0$  and putting  $\delta = 1$ .
- The equations of a coupled theory of local thermoelasticity with temperature-dependent mechanical properties result from (1),



- (8), in the limiting case  $\alpha \rightarrow 0$  and (11) by letting  $\tau_0 = \xi = 0$  and putting  $\delta = 1$ .
- The equations of a coupled theory of nonlocal thermoelasticity with temperature-independent mechanical properties result from (1), (8) and (9), in the limiting case  $\alpha \rightarrow 1$  and (11) by letting  $\tau_0 = 0$ ,  $F(T_0) = 1$  and putting  $\delta = 1$ .
  - The equations of a coupled theory of local thermoelasticity with temperature-independent mechanical properties result from (1), (8) and (9), in the limiting case  $\alpha \rightarrow 1$  by letting  $\tau_0 = \xi = 0$ ,  $F(T_0) = 1$  and putting  $\delta = 1$ .
  - The equations of a generalized theory of nonlocal thermoelasticity with temperature-dependent mechanical properties without fractional derivatives are obtained from Eqs. (1), (8) and (9) by letting the fractional parameter  $\alpha \rightarrow 1$  and putting  $\delta = 1$ .
  - The equations of a generalized theory of nonlocal thermoelasticity with temperature-independent mechanical properties without fractional derivatives are obtained from Eqs. (1), (8) and (9) by letting the fractional parameter  $\alpha \rightarrow 1$  and putting  $\delta = 1$  and  $F(T_0) = 1$ .
  - The equations of the generalized nonlocal thermoelasticity without energy dissipation with temperature-dependent mechanical properties without fractional derivatives result from Eqs. (1), (8) and (9) by letting  $\tau_0 = 1$ ,  $\alpha \rightarrow 1$  and  $\delta = 0$ .

## Numerical results

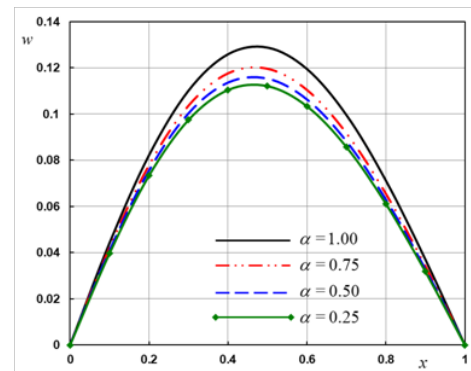
In the present work, the thermoelastic coupling effect is analyzed by considering a beam made of silicon (Table 1).

**Table 1** Mechanical and thermoelastic properties parameter of the graded nanobeam ( $T_0 = 293K$ )

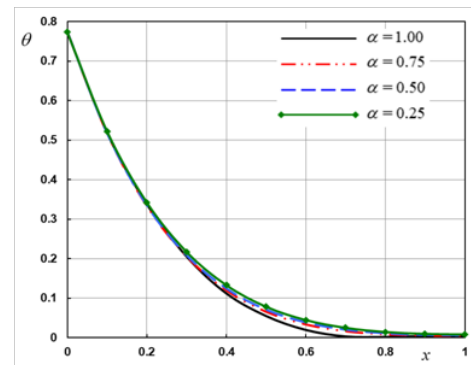
Material properties/ Material	Metal (Aluminum)
Thermal conductivity $K$ ( $Wm^{-1}K^{-1}$ )	156
Young's modulus $E$ (GPa)	169
Density $\rho$ ( $Kgm^{-3}$ )	2330
Thermal expansion $\alpha_t$ ( $K^{-1}$ )	$2.59 \times 10^{-6}$
Thermal diffusivity $k$ ( $m^2s^{-1}$ )	$84.18 \times 10^{-6}$
Poisson's ratio $\nu$	0.22
Specific heat $C_E$ (J/kgK)	713

Also, the length-to-thickness ratio of the nanobeam is fixed as  $L/h = 10$  and the dimensional parameters carried out in the numerical simulations are  $L = 1$  and  $z = h/3$ . For all numerical calculations *Mathematica* programming Language has been used. The numerical calculations of the flexure moment  $M$  thermodynamical temperature  $\theta$ , displacement  $u$ , lateral vibration  $w$  and strain  $e$  have been considered for various values of the nonlocal parameter  $\xi = \xi \times 10^{-6}$  fractional order  $\alpha$  thickness of nanobeam and periodic frequency  $\omega$ . The results are investigated graphically in Figures 2–17 in the wide range of  $0 \leq x \leq 1.0$  at different positions of  $x$ . Some plots

consider the present quantities through the length of the beam and others take into account both the length and thickness directions.



**Figure 2** The transverse deflection  $w$  with different fractional order parameter  $\alpha$ .



**Figure 3** The temperature  $\theta$  with different fractional order parameter  $\alpha$ .

From the results we obtained it is clear that, the field quantities such as the deflection, temperature, displacement, and bending moment distributions depend not only on the time  $t$  and space coordinate, but also depend on the fractional parameter  $\alpha$ , nonlocal parameter  $\xi$ , ramping time parameter  $t_0$  and effect of reference temperature  $F(T_0) = 1 - \beta T_0$ . Numerical calculations and graphs have been divided into four cases. In this work, the obtained results have been compared with the results obtained previously in.<sup>22,23,35,40</sup>

## The effect of the fractional order parameter

In the first case, we represent the dimensionless lateral vibration, thermodynamic temperature, displacement, and bending moment at different fractional order parameter  $\alpha$  to stand on the effect of this parameter on all the studied fields (Figures 2–5). The computations were carried out for wide range of  $x$  ( $0 \leq x \leq 1.0$ ), for different values of the parameter  $\alpha$  with wide range ( $0 < \alpha \leq 1.0$ ) when  $t_0 = 0.1$ ,  $\tau_0 = 0.2$ , and  $\xi = 1$ . The different values of the parameter  $\alpha$ , describes two types of conductivity (weak conductivity, ( $0 < \alpha < 1.0$ ), normal conductivity,  $\alpha = 1$ ), respectively. From the different figures it is observed that the nature of variations of all the field variables for fractional order parameter is significantly different. The difference is more prominent for higher values of fractional derivative  $\alpha$ . This observation is consistent with previous results in.<sup>22</sup> We also observe the following important facts from Figures 3–6.

- The lateral vibration  $w$  distribution has large change against to fractional order parameter  $\alpha$ . Also, the lateral vibration increases with the increase of the parameter  $\alpha$ .

- b. From figure 1, lateral vibration at  $x = 0, L$  is zero as shown, which agrees with the boundary condition prescribed. This coincided with the mechanical boundary condition that the two ends of the nanobeam are clamped.
- c. We observed that at the lateral vibration  $w$  reaches a maximum value at some distance from the surface of the nanobeam and then it decreases with the increase of distance  $x$ .
- d. The temperature distribution  $\theta$  have small differences for different values of  $\alpha$ .
- e. Figure 3 indicate that the  $e$  temperature has a maximum value at boundary  $x = 0$  and decreases smoothly and finally tends to zero after some distance (which satisfy the boundary conditions).
- f. Figure 3 shows that the displacement  $u$  increases as  $x$  increases. It can be observed that the fractional order parameter  $\alpha$  has a great effects on the displacement distribution.
- g. From Figure 4 it is observed that the bending moment  $M$  increases as the distance  $x$  increases and then it decreases with the increase of distance  $x$ . Figure 5 shows that the bending moment larg dependent on the variation of the fractional order parameter  $\alpha$ .
- h. According to these results, we have to construct a new classification to all the materials according to its fractional parameter, where this parameter becomes new indicator of its ability to conduct the thermal energy; confirms the results obtained in.<sup>22</sup>

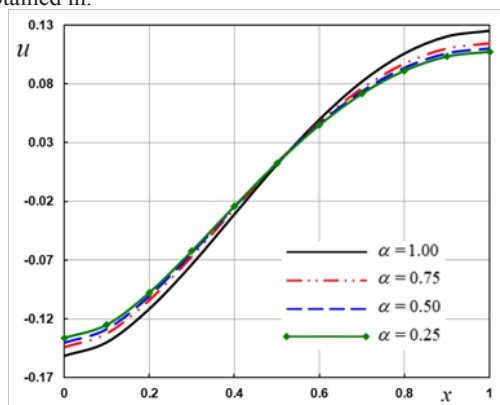


Figure 4 The displacement  $u$  with different fractional order parameter  $\alpha$ .

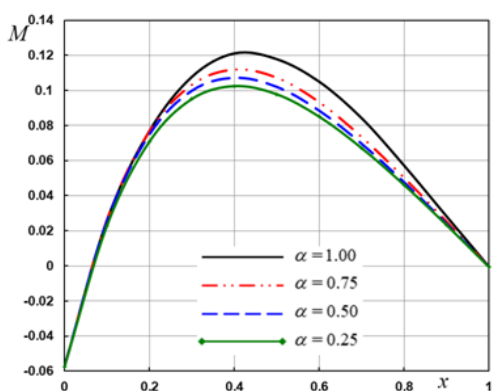


Figure 5 The flexure moment  $M$  with different fractional order parameter  $\alpha$ .

## The effect of the nonlocal parameter

Figures 6–9 describe the temperature, displacement, lateral vibration and bending moment field distributions for different values of nonlocal parameter  $\bar{\xi}$ . In this case, we notice that when the nonlocal parameter  $\bar{\xi}$  vanishing  $\bar{\xi} = 0$  indicates the old situation (local model of elasticity) while other values indicate the nonlocal theories of elasticity and thermo elasticity. The figures 6–9 show that this parameter has significant effect on all the fields. The waves reach the steady state depending on the value of the nonlocal parameter  $\bar{\xi}$ . The conclusion remarks from the Figures can be shortened as follows:

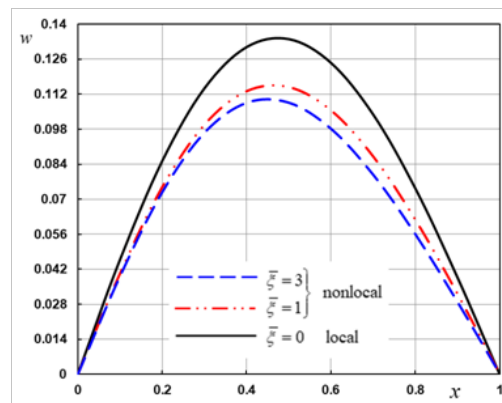


Figure 6 The transverse deflection  $w$  with different nonlocal parameter  $\bar{\xi}$ .

- a. The lateral vibration  $w$  decrease when the value of  $\bar{\xi}$  increases while the values of the bending moment  $M$  increases when the value of  $\bar{\xi}$  increases.
- b. It can be seen from Figure 7 that the temperature  $\theta$  small dependent on the variation of nonlocal parameter  $\bar{\xi}$ .
- c. From Figure 5, the values of the displacement  $u$  start increasing with the nonlocal parameter in the range  $0 \leq x \leq 0.6$ , thereafter increasing to maximum amplitudes in the range  $0.6 \leq x \leq 1.0$ . This shows the difference between the local generalized thermoelasticity and the nonlocal generalized thermoelasticity models.
- d. It can be seen from Figure 9 that the bending moment larg dependent on the variation of nonlocal parameter  $\bar{\xi}$ .
- e. As in,<sup>23,35,40</sup> the field variables have pronounced sensitive to the variation of the nonlocal parameter.

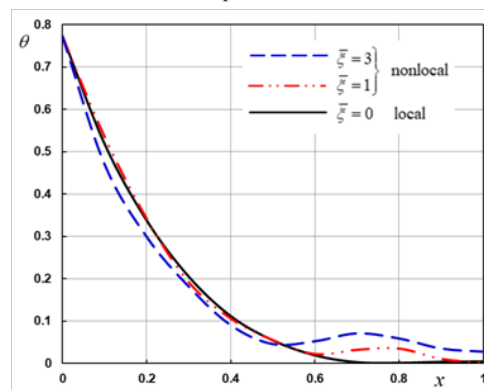


Figure 7 The temperature  $\theta$  with different nonlocal parameter  $\bar{\xi}$ .

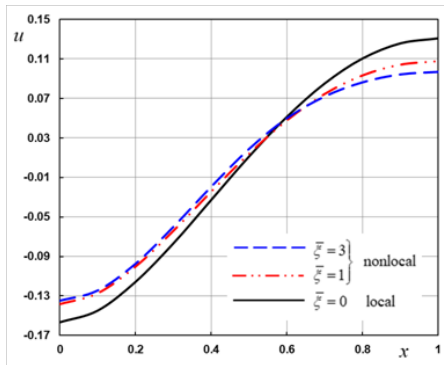


Figure 8 The displacement  $u$  with different nonlocal parameter  $\bar{\xi}$ .

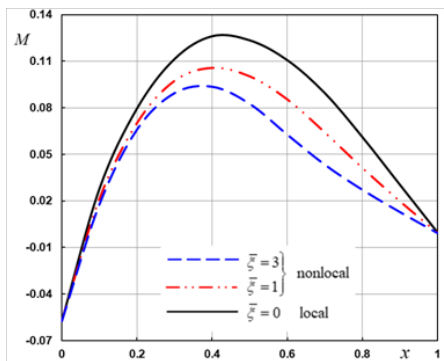


Figure 9 The flexure moment  $M$  with different nonlocal parameter  $\bar{\xi}$ .

### The effect of the ramping time parameter

This case illustrating how the field quantities vary with the different values of the ramping time parameter  $t_0$  with constant  $\alpha = 0.7$ ,  $\tau_0 = 0.2$ , and  $\bar{\xi} = 1$ . The numerical results are obtained and presented graphically in Figures 10–13. We can see the significant effect of the ramping time parameter  $t_0$  on all the studied fields. Also, we can conclude that:

- The increasing in the value of the ramping time parameter  $t_0$  causes decreasing in the values of the lateral vibration  $w$ , temperature  $\theta$  and bending moment  $M$  which is very obvious in the peak points of the curves.
- Also, the values of the displacement  $u$  start decreasing with the ramping time parameter in the range  $0 \leq x \leq 0.6$ , thereafter increasing to maximum amplitudes in the range  $0.6 \leq x \leq 1.0$ .
- It can be observed that the ramping time parameter  $t_0$  has great effects on the distribution of field quantities. This is also consistent with what has been obtained in.<sup>40</sup>

### The effect of the temperature-independent modulus

This case presents the effect of temperature-dependent properties (reference temperature) on the field variables for fixed values of the ramping time parameter  $t_0 = 0.1$  and the nonlocal parameter  $\bar{\xi} = 1$  according to the equation  $F(T_0) = 1 - \beta T_0$ . For a comparison of the results, the lateral vibration, the temperature, the displacement, and the bending moment of the nano-beam are shown in Figures 14–17. We found that, the increasing in the value of  $F(T_0)$  causes increasing in the values of the lateral vibration, temperature, displacement fields which is very obvious in the peak points of the curves. When the

modulus of elasticity is a linear function of reference temperature ( $F(T_0) = 0.5, 0.8$ ) and in the case of a temperature-independent modulus ( $\beta = 0$ ). On the other hand, high sensitivity is observed for the effect of the reference temperature on all of the studied variables. This also agrees with what Zenkour has reached in.<sup>40</sup>

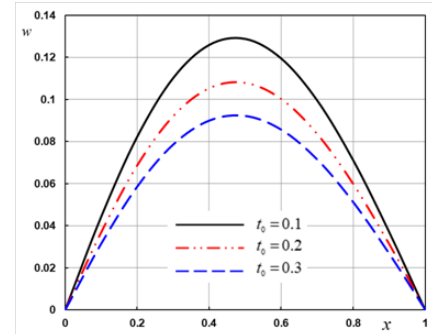


Figure 10 The transverse deflection  $w$  with different ramping time  $t_0$ .

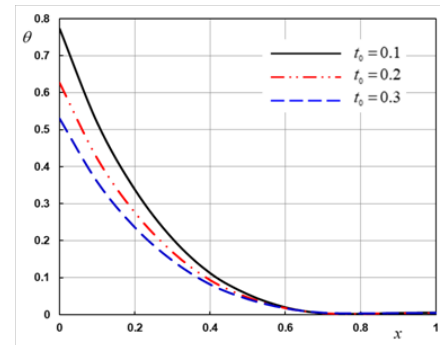


Figure 11 The temperature  $\theta$  with different ramping time  $t_0$ .

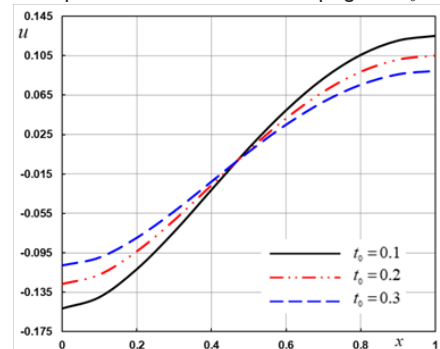


Figure 12 The displacement  $u$  with different ramping time  $t_0$ .

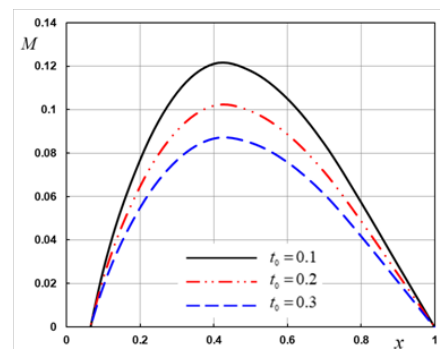
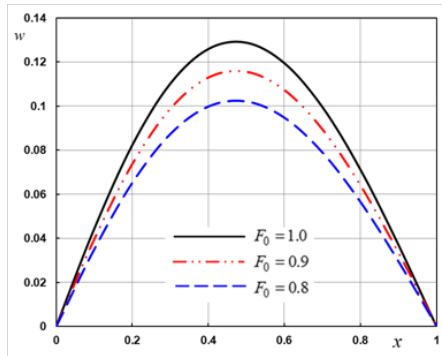
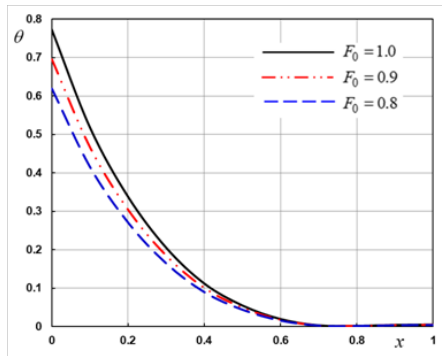


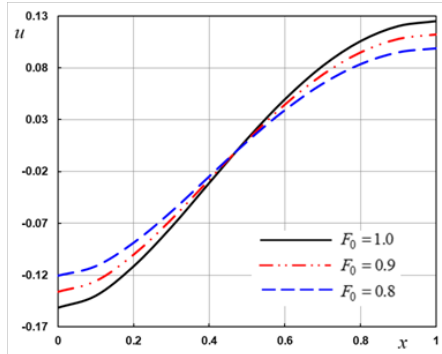
Figure 13 The flexure moment  $M$  with different ramping time  $t_0$ .



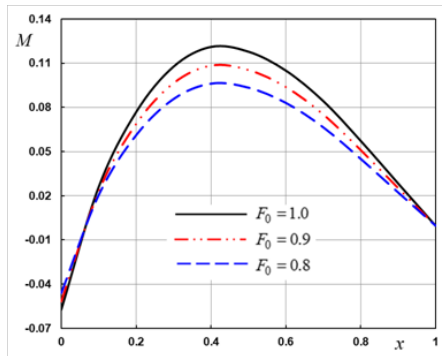
**Figure 14** The transverse deflection  $W$  in the case of temperature-independent properties.



**Figure 15** The temperature  $\theta$  in the case of temperature-independent properties.



**Figure 16** The displacement  $u$  in the case of temperature-independent properties.



**Figure 17** The flexure moment  $M$  in the case of temperature-independent properties.

## Conclusion

The thermal and mechanical properties of materials vary with temperature, so that the temperature dependent on material properties must be taken into consideration in the thermal stress analysis of these elements. In this work, a new model of nonlocal generalized thermoelasticity based upon Euler–Bernoulli theory for nanobeams is constructed. The vibration characteristics of the deflection, temperature, displacement and bending moment of nanobeam subjected to ramp-type heating are investigated. The effects of the nonlocal parameter  $\bar{\xi}$ , fractional order parameter  $\alpha$  and the ramping time parameter  $t_0$  on the field variables are investigated. Numerical technique based on the Laplace transformation has been used. The effects of the nonlocal parameter, reference temperature and the ramping time parameter on all the studied field quantities have been shown and presented graphically.

According to the results shown in all figures, it is find that:

- The nonlocal parameter  $\bar{\xi}$  has significant effects on all the studied fields.
- The effect of the fractional parameter on all the studied fields is very significant.
- The thermo elastic stresses, displacement and temperature have a strong dependency on the ramping time parameter.
- The parameter  $\alpha$  and the dimension cause significant changes on all the studied fields and the phenomenon of finite speeds of propagation are manifested in all figures.
- New classification of the materials must be constructed according to the fractional parameter which describes the ability of the material to conduct the heat.
- This study is very important for micro scale problems because in these cases the material parameters are temperature dependent.

$\lambda, \mu$	Lam'e's constants	$K$	thermal conductivity
$\alpha_t$	thermal expansion coefficient	$\alpha$	Fractional order parameter
$\gamma$	coupling parameter	$q_i$	components of the heat flows vector
$T_0$	environmental temperature	$\delta_{ij}$	Kronecker's delta function
$\theta = T - T_0$	temperature increment	$u_i$	displacement components
$T$	absolute temperature	$F_i$	body force components
$C_E$	specific heat	$Q$	heat source
$e$	cubical dilatation	$\tau_0$	relaxation time
$\sigma_{ij}$	nonlocal stress tensor	$h$	nanobeam thickness
$e_{ij}$	strain tensor	$\rho$	material density
$L$	nanobeam length	$b$	nanobeam width
$A = bh$	cross-section area	$oxyz$	Cartesian coordinate
$\tau_{ij}$	local stress tensor	$\nabla^2$	Laplacian operator
$\bar{\xi}$	Nonlocal parameter	$E$	Young's modulus
$\Gamma(\alpha)$	Gamma function	$t$	the time
$\Gamma(\alpha)$	Gamma function		
$\nu$	Poisson's ratio		



## Acknowledgements

None

## Conflict of interest

The author declares that there is no conflict of interest.

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