Determination of parametric average code length of inaccuracy measure

Abstract

In this paper, a parametric generalized inaccuracy measures $I_i^x(P:Q)$ with a two-parametric generalized average codeword length $L_i^x(P)$ has been proposed. It is shown that these measures are the generalizations of some well-known measures that already exist in the literature of information theory. Further, the bounds for the noiseless coding theorems for a discrete noiseless channel have been developed. The noiseless coding theorems for discrete channel proved in this paper are verified by well known coding schemes (Huffman and Shannon-Fano coding schemes) using empirical data.

Keywords: average codeword length, entropy, holder's inequality, huffman codes, inaccuracy measure, noiseless coding theorem, shannon-fano code

Ams subject classification: 94A17, 94A24, 26D15.

Introduction

From past three decades, entropy which is branch of statistical sciences has been used to determine the degree of variability, describes how uncertainty should be quantified in a skillful manner for representation. Statistical entropy has some conflicting explanations so that sometimes it measures two complementary conceptions like information and lack of information. Claude Shannon through two outstanding contributions in 1948 and 1949 relates it with positive information. These were followed by a flood of research papers hypothesize upon the possible applications in almost every field such as pure mathematics, semantics, physics, management, thermodynamics, botany, econometrics, operations research, psychology, epidemiological studies, disease management and related disciplines. Information theory has also had an important role in shaping theories of perception, cognition, and neural computation. When the message is readily measurable, we can say that the information is the reduction of uncertainty. But we usually encountered lossy information i.e a part of the transmitted information reaches the destination in a distorted form. In statistical theory of information, certain specialized terms which need to be translated into a measurable form. A source is similar to the space of a random experiment. A finite sequence of characters is called a word in the same way that the sequence of a number of outcomes associated with the repetition of an experiment may be designated as an event. An interesting observation can be made about the entropy of a binary source. Binary coding offers an interesting practical opportunity for encoding.

Encoding techniques provide a most direct application in the reduction of information. It is one of the best applications of information measure in noiseless coding theorem which gives the limits for suitable encoding of information in terms of information measure. Thus, we find the minimum value of a mean codeword length subject to a given constraint on codeword lengths. However, since the codeword lengths are integers, the minimum value will lie between two limits and a noiseless coding theorem seeks to find these two limits which are in terms of some measure of entropy for a given mean and a given constraint.

The most important kinds of codes are those that do not require spacing. Codes with this property are known as uniquely decipherable codes. The two basic theorems of coding theory under the condition of uniquely decipherability was investigated by several authors; Nath,1 Longo,7 Belis & Guiasu,7 Guiasu & Picard,4 Gurdial & Pessoa,5 extended the theorem by finding lower bounds for useful mean codeword length of order $a$; also various authors like Jain & Tuteja,6 Taneja et al.,7 Bhatia,4 Hooda & Bhaker,9,10 Khan et al.,11 Baig & Arif,12–14 have studied generalized coding theorems by considering different generalized ‘useful’ information measures under the condition of unique decipherability. Coding theorems allows us to illustrate the significance of these theorems.

Preliminaries

Suppose we have an event $X$, where $x$ represents a particular outcome of the event and $p_i$ be the probability of outcome $x_i$, then according to Shannon the self information of $x_i$ is defined as

$$I(x_i) = \log \frac{1}{p_i} = -\log p_i \quad (2.1)$$

Let $X$ be a discrete random variable taking values $X = (x_1,x_2,...,x_n)$ with respective probabilities $P = (p_1,p_2,...,p_n)$, $p_i \geq 0 \forall i = 1,2,...,n$ and $\sum_{i=1}^{n} p_i = 1$. Shannon15 gives the following measure of information and call it entropy.

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \quad (2.2)$$

The measure (2.2) serves as a suitable measure of entropy. Let $p_1,p_2,...,p_n$ be the probabilities of $n$ codewords to be transmitted and let their lengths $l_1,l_2,...,l_n$ satisfy Kraft (1949) inequality,

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \quad (2.3)$$

Where $D$ is the size of code alphabet.

For uniquely decipherable codes, Shannon15 showed that for all codes satisfying (2.3), the lower bound of the mean codeword length,

$$L = \sum_{i=1}^{n} p_i l_i \quad (2.4)$$

lies between $H(P)$ and $H(P) + 1$.
Campbell\(^{15}\) considered the more general exponentiated mean code word length as
\[
L_\alpha = \frac{\alpha}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i D \left( \frac{\alpha}{1 - \alpha} \right) D \left( p_i q_i^{\alpha - 1} \right) \right), \quad 0 < \alpha, \alpha \neq 1
\]  
(2.5)
and showed that subject to (2.3), the minimum value of (2.5) lies between \(R_\alpha(P)\) and \(R_\alpha(P) + 1\), where
\[
R_\alpha(P) = \frac{1}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i^\alpha \right), \quad 0 < \alpha, \alpha \neq 1
\]  
(2.6)
is Renyi's\(^{13}\) entropy of order \(\alpha\).

**Inaccuracy measure**

Suppose that an experimenter states the probabilities of the various possible outcomes of an experiment. This statement can lack precision in two ways: first it may not have enough information and so the statement is vague, or some of the information may be incorrect. All statistical inference related problems are concerned with making statements which may be inaccurate in either or both of these ways. Kerridge\(^{18}\) proposed the inaccuracy measure that can take accounts for these two types of errors.

Let us consider that the probability of the \(i\)th event is \(q_i\), when the true probability is \(p_i\). Then the inaccuracy of the investigator, as proposed by Kerridge\(^{18}\) can be measured as:
\[
I(P,Q) = -\sum_{i=1}^{n} p_i \log q_i
\]  
(3.1)
Where \(P=(p_1, p_2, \ldots, p_n)\) and \(Q=(q_1, q_2, \ldots, q_n)\) are two discrete probability distributions such that \(p_i \geq 0, q_i \geq 0\) and \(\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1\).

**Generalized inaccuracy in terms of noiseless coding theorems**

The major objective of this portion is to give a detailed statement and proof of the discrete noiseless theorems. Let us define two parametric generalized inaccuracy measures \(I_\alpha^\beta(P,Q)\) and a two parametric generalized average codeword length \(L_\alpha^\beta(P)\). The proof of the following theorems has been described and has shown that these measures are the generalizations of some known measures but has more flexibility. The aim is to develop a measure of information of a discrete system. At the end, the lower and upper limits of \(I_\alpha^\beta(P)\) in terms of \(I_\alpha^\beta(P,Q)\) has obtained.

Let us introduce the parametric generalized inaccuracy measure as:
\[
I_\alpha^\beta(P,Q) = \frac{\beta}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i^\beta q_i^{\alpha - 1} \right)
\]  
(4.1)
Where,
\[
0 < \alpha < 1, 0 < \beta \leq 1, \quad p_i \geq 0, q_i \geq 0, \forall i = 1, 2, \ldots, n, \sum_{i=1}^{n} p_i \sum_{i=1}^{n} q_i = 1.
\]

\(\alpha\) and \(\beta\) - discrepancies can provide more robust solutions with respect to outliers and additive noise which improve accuracy. The following are the particular cases of the measure (4.1)

**Particular cases for (4.1):**

I. When \(\beta = 1\), (4.1) reduces to the inaccuracy measure given by Nath\(^{1}\), i.e.,
\[
I_\alpha(P,Q) = \frac{1}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i^\beta q_i^{\alpha - 1} \right)
\]
II. When \(\beta = 1\) and \(\alpha \rightarrow 1\), (4.1) reduces to inaccuracy measure due to Kerridge\(^{18}\), i.e.,
\[
I(P,Q) = -\sum_{i=1}^{n} p_i \log q_i
\]
III. When \(p_i = q_i\forall i = 1, 2, \ldots, n\), then (4.1) reduces to information measure given by Bhat & Baig,\(^{12-14}\) i.e.,
\[
H_\alpha^\beta(P) = \frac{\beta}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i^{\beta \alpha - 1} \right)
\]
IV. When \(\beta = 1\) and \(p_i = q_i\forall i = 1, 2, \ldots, n\), then (4.1) reduces to Renyi\(^{17}\) entropy, i.e.,
\[
H_\alpha^\beta(P) = \frac{1}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i^{\alpha - 1} \right)
\]
V. When \(\beta = 1\), \(\alpha \rightarrow 1\) and \(p_i = q_i\forall i = 1, 2, \ldots, n\), then (4.1) reduces to Shannon\(^{15}\) entropy, i.e.,
\[
H_\alpha^\beta(P) = \frac{1}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i \right)
\]
VI. When \(\alpha \rightarrow 1, \beta = 1\) and \(p_i = q_i = \frac{1}{n}\forall i = 1, 2, \ldots, n\), then (4.1) reduces to maximum entropy, i.e.,
\[
H_\alpha^\beta(P) = \log_D(n)
\]
Further we define a new generalized code-word length of order \(\alpha\) and type \(\beta\) as:
\[
L_\alpha^\beta(P) = \frac{\alpha}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i D \left( \frac{\alpha}{1 - \alpha} \right) \right), \quad 0 < \alpha < 1, 0 < \beta \leq 1
\]  
(4.2)
Where, \(D\) is the size of code alphabet.

**Particular cases for (4.2):**

I. For \(\beta = 1, \) (4.2) reduces to codeword length given by Campbell\(^{16}\), i.e.,
\[
L_\alpha(P) = \frac{\alpha}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i \right)
\]
II. For \(\beta = 1\) and \(\alpha \rightarrow 1\), (2.4) reduces to the optimal code-word length identical to Shannon\(^{15}\), i.e.,
\[
L_\alpha = \frac{1}{1 - \alpha} \log_D \left( \sum_{i=1}^{n} p_i \right)
\]
III. For \(\beta = 1\) and \(\alpha = 1\), (2.4) reduces to 1. i.e., \(L_\alpha = 1\)

Now we found the bounds of (4.2) in terms of (4.1) under the condition
\[
\sum_{i=1}^{n} p_i^\beta q_i^{\beta - \beta D^{-1}} \leq 1
\]  
(4.3)
This is generalization of Kraft’s inequality. It is easy to see that when \( P_i = q_i \forall i = 1, 2, \ldots, n \), then the inequality (4.3) reduces to Kraft’s (1949) inequality (2.3), where \( D \) is the size of code alphabet.

**Theorem 4.1:** For all integers \( D > 1 \) the code word lengths \( l_i, l_2, \ldots, l_n \) satisfies the condition (4.3), then the inequality

\[
L_d(\alpha) \geq L_d(\alpha, \beta, p) \geq L_d(\alpha, \beta, q) \geq 0, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1, \quad \text{(4.4)}
\]

is fulfilled. Where \( L_d(\alpha) \) and \( L_d(\alpha, \beta, p) \) are defined in (4.1) and (4.2) respectively. Furthermore, the equality holds if and only if

\[
l_i = -\log_D \left[ \sum_{i=1}^{n} p_i \rho q_i \rho\{{\alpha}[i-1]\} \right]
\]

\[
(4.5)
\]

**Proof:** By Holder’s inequality, we have

\[
\left( \sum_{i=1}^{n} x_i \rho \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^{n} y_i \rho \right)^{\frac{1}{\beta}} \leq \sum_{i=1}^{n} x_i y_i \rho
\]

\[
(4.6)
\]

or equivalently the above equation can be written as

\[
\sum_{i=1}^{n} x_i \rho \left( y_i \rho \right)^{\frac{1}{\beta}} \leq \left( \sum_{i=1}^{n} x_i \rho \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^{n} y_i \rho \right)^{\frac{1}{\beta}}
\]

\[
(4.7)
\]

Making the substitution

\[
x_i = p_i \rho^{\frac{1}{\alpha}} D^{-1}, \quad y_i = p_i \rho^{\frac{1}{\beta}} q_i \rho \}
\]

\[
(4.8)
\]

Using these values in (4.6) we get,

\[
\sum_{i=1}^{n} p_i \rho^{\frac{1}{\alpha}} D^{-1} \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}} \leq \sum_{i=1}^{n} p_i \rho q_i \rho \}
\]

\[
(4.9)
\]

Now using the inequality (4.3), we get

\[
\sum_{i=1}^{n} p_i \rho D^{-1} \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}} \leq 1
\]

\[
(4.10)
\]

Or equation (4.9) can be written as

\[
\sum_{i=1}^{n} p_i \rho D^{-1} \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}} \leq 1
\]

\[
(4.11)
\]

Taking logarithms to both sides with base \( D \) to equation (4.10), we get

\[
\frac{\alpha}{\alpha - 1} \log_D \left[ \sum_{i=1}^{n} p_i \rho D^{-1} \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}} \right] \leq 1
\]

\[
(4.12)
\]

Or equivalently we can write the equation (4.11), as

\[
\alpha \log_D \left[ \sum_{i=1}^{n} p_i \rho D^{-1} \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}} \right] \geq 1
\]

\[
(4.13)
\]

As \( 0 < \beta \leq 1 \), multiply equation (4.12) both sides by \( \beta \), we get

\[
\frac{\alpha \beta}{\alpha - 1} \log_D \left[ \sum_{i=1}^{n} p_i \rho D^{-1} \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}} \right] \geq \beta
\]

\[
(4.14)
\]

This implies

\[
L_d(\alpha) \geq L_d(\alpha, \beta, p) \geq L_d(\alpha, \beta, q) \geq 0, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1,
\]

\[
(4.15)
\]

Now we will show that the equality in (4.4) holds if and only if

\[
l_i = -\log_D \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right] \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1.
\]

\[
(4.16)
\]

Or equivalently, we can write the equation (4.14) as

\[
D^{-1} = \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}}
\]

\[
(4.17)
\]

Raising both sides to the power \( \alpha^{-1} \), to equation (4.15), and after suitable simplification we get

\[
\sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} = \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}}
\]

\[
(4.18)
\]

This implies that

\[
L_d(\alpha) = L_d(\alpha, \beta, p) = L_d(\alpha, \beta, q) \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1.
\]

\[
(4.19)
\]

Proof: For every code with lengths \( l_1, l_2, \ldots, l_n \) satisfies the condition (4.3), \( L_d(\alpha, p) \) can be made to satisfy the inequality

\[
L_d(\alpha) \geq L_d(\alpha, \beta, p) \geq L_d(\alpha, \beta, q) \geq 0, \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1.
\]

\[
(4.20)
\]

Or equivalently the above equation can be written as

\[
\sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} = \left[ \sum_{i=1}^{n} p_i \rho q_i \rho \{{\alpha}[i-1]\} \right]^{\frac{1}{\alpha}}
\]

\[
(4.21)
\]

This implies

\[
L_d(\alpha) \geq L_d(\alpha, \beta, p) \geq L_d(\alpha, \beta, q) \quad 0 < \alpha < 1, \quad 0 \leq \beta \leq 1.
\]

\[
(4.22)
\]
Determination of parametric average code length of inaccuracy measure

Now we choose the code-word lengths \( l_i, l_2, \ldots, l_n \) in such a way that they satisfy the inequality,

\[
-\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) \leq l_i < -\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) + 1
\]

\( \text{Consider the interval } \delta_i = \left[ -\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right), -\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) + 1 \right], \text{ of length unity. In every } \delta_i, \text{ there lies exactly one positive integer } l_i \text{ such that,}
\]

\[
0 < -\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) \leq l_i < -\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) + 1 \quad \text{(4.20)}
\]

Now we will first show that the sequence \( l_i, l_2, \ldots, l_n \) thus defined satisfies the inequality (4.3), which is generalization of Kraft’s (1949) inequality.

From the left inequality of (4.20), we have

\[
-\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta (l_i-1)} \right) \leq l_i
\]

Or equivalently, the above equation can be written as

\[
D^{-\frac{1}{\alpha}} \leq \frac{q_i^{\alpha}}{\sum_{i=1}^n p_i^\beta q_i^{\beta (l_i-1)}} \quad \text{(4.21)}
\]

Multiply equation (4.21) both sides by \( p_i^\beta q_i^{\beta l_i} \) then summing over \( i = 1, 2, \ldots, n \) both sides to the resulted expression, and after suitable operations, we get the required result (4.3), i.e.,

\[
\frac{1}{\alpha} \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \leq 1
\]

Now the last inequality of (4.20), gives

\[
l_i < -\log q_i^\alpha + \log D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) + 1
\]

Or equivalently, the above equation can be written as

\[
D^{l_i} < q_i^{\alpha} \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) D \quad \text{(4.22)}
\]

As \( 0 < \alpha < 1, \text{ then } (1-\alpha) > 0, \text{ and } \left( \frac{1-\alpha}{\alpha} \right) > 0, \text{ raising both sides to the power } \left( \frac{1-\alpha}{\alpha} \right) > 0, \text{ to equation (4.22), and after suitable operations, we get}
\]

\[
D^{-\frac{1}{\alpha}} < q_i^{(\alpha-1)} \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) \frac{1}{\alpha} \frac{1-\alpha}{D} \quad \text{(4.23)}
\]

Multiply equation (4.23) both sides by \( p_i^\beta q_i^{\beta l_i} \) then summing over \( i = 1, 2, \ldots, n \) both sides to the resulted expression and after suitable simplification, we get

\[
\sum_{i=1}^n p_i^\beta D^{-\frac{1}{\alpha}} \left( q_i^{(\alpha-1)} \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) \right) \leq \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \frac{1}{\alpha} \frac{1-\alpha}{D} \quad \text{(4.24)}
\]

Or equivalently, the above equation can be written as

\[
\sum_{i=1}^n p_i^\beta D^{-\frac{1}{\alpha}} \left( q_i^{(\alpha-1)} \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) \right) \leq \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \frac{1}{\alpha} \frac{1-\alpha}{D} \quad \text{(4.25)}
\]

Taking logarithms with base \( D \) both sides to the equation (4.24), we get

\[
\log_D \left( \sum_{i=1}^n p_i^\beta D^{-\frac{1}{\alpha}} \right) < \log_D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) \frac{1}{\alpha} \frac{1-\alpha}{D} \quad \text{(4.26)}
\]

As \( 0 < \alpha < 1, 0 < \beta \leq 1 \text{ then } (1-\alpha) > 0 \text{ and } \frac{\alpha \beta}{1-\alpha} > 0 \), multiply equation (4.25), both sides by \( \frac{\alpha \beta}{1-\alpha} > 0 \) we get

\[
\frac{\alpha \beta}{1-\alpha} \log_D \left( \sum_{i=1}^n p_i^\beta D^{-\frac{1}{\alpha}} \right) < \beta \log_D \left( \sum_{i=1}^n p_i^\beta q_i^{\beta l_i} \right) \quad \text{(4.27)}
\]

This implies that

\[
I_0^\beta (P) < I_0^\beta (P; Q) + \beta. \text{ Hence the result for } 0 < \alpha < 1, 0 < \beta \leq 1 \text{.}
\]

Thus from above two coding theorems, we have shown that

\[
I_0^\beta (P) < I_0^\beta (P; Q) + \beta \where 0 < \alpha < 1, 0 < \beta \leq 1 \text{.}
\]

Exemplifying the proposed generalization

Here we demonstrate the accuracy of the above theorems 4.1 and 4.2 by taking speculative information. Let us suppose a discrete random variable \( X \) taking fixed values as:

\[
X = \{ x_1, x_2, x_3, x_4, x_5, x_6 \}
\]

Suppose that the experimenter asserts the probability distribution of this random variable as:

\[
Q = (q_1, q_2, q_3, q_4, q_5, q_6) = (0.4, 0.2, 0.15, 0.12, 0.08, 0.05)
\]

Whereas the true probability distribution of this random variable is:

\[
P = (p_1, p_2, p_3, p_4, p_5, p_6) = (0.41, 0.18, 0.15, 0.13, 0.1, 0.03)
\]

Now using Huffman coding scheme the values of \( I_0^\beta (P; Q), I_0^\beta (P; Q) + \beta \) and \( \gamma \) for different values of \( \alpha \) and \( \beta \) are shown in the following table: (Table 1) (Table 2)

**Conclusion**

In this work, a new class of generalized inaccuracy measure with a generalized mean code word length depending on the parameters \( \alpha \) and \( \beta \) has been introduced. The limits (upper and lower) of the proposed mean codeword length has also been established which approximates the measure of accuracy. By using Shannon-Fano coding and Huffman coding schemes theorems 4.1 and 4.2 holds in both the cases of i.e.
\[ I_\alpha^\beta(P:Q) \leq I_\alpha^\beta(P) < I_\alpha^\beta(P:Q) + \beta \] where \( 0 < \alpha < 1, 0 < \beta \leq 1. \)

The proposed generalized mean codeword length has less codeword length in case of Huffman coding scheme as compared to using Shannon-Fano coding scheme. Also by using Huffman coding scheme and Shannon-Fano coding scheme, the efficiency \( (\eta) \) of the proposed generalized mean codeword length is greater in case of Huffman coding scheme as compared to using Shannon-Fano coding scheme.

### Table 1 Values \( I_\alpha^\beta(P:Q), I_\alpha^\beta(P:Q) + \beta, \) and for different values of \( \alpha \) and \( \beta \) using Huffman coding scheme

<table>
<thead>
<tr>
<th>( p_i )</th>
<th>( q_i )</th>
<th>Huffman Codeword's</th>
<th>( l_i )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( I_\alpha^\beta(P:Q) )</th>
<th>( I_\alpha^\beta(P) )</th>
<th>( \eta = \frac{I_\alpha^\beta(P:Q)}{I_\alpha^\beta(P)} \times 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.41</td>
<td>0.4</td>
<td>1</td>
<td>1</td>
<td>0.9</td>
<td>1</td>
<td>2.29</td>
<td>2.3596</td>
<td>97.05%</td>
</tr>
<tr>
<td>0.18</td>
<td>0.2</td>
<td>0</td>
<td>3</td>
<td>0.9</td>
<td>0.9</td>
<td>3.95</td>
<td>4.031</td>
<td>97.99%</td>
</tr>
<tr>
<td>0.15</td>
<td>0.15</td>
<td>1</td>
<td>3</td>
<td>0.8</td>
<td>1</td>
<td>2.3182</td>
<td>2.4208</td>
<td>95.76%</td>
</tr>
</tbody>
</table>

Now using Shannon-Fano coding scheme the values of \( I_\alpha^\beta(P:Q), I_\alpha^\beta(P:Q) + \beta, \) and \( \eta \) for different values of \( \alpha \) and \( \beta \) are shown in the following table:

### Table 2 Values \( I_\alpha^\beta(P:Q), I_\alpha^\beta(P:Q) + \beta, \) and for different values of \( \alpha \) and \( \beta \) using Shannon-Fano coding scheme

<table>
<thead>
<tr>
<th>( p_i )</th>
<th>( l_i )</th>
<th>Shannon-Fano codewords</th>
<th>( l_i )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( I_\alpha^\beta(P:Q) )</th>
<th>( I_\alpha^\beta(P) )</th>
<th>( \eta = \frac{I_\alpha^\beta(P:Q)}{I_\alpha^\beta(P)} \times 100 )</th>
</tr>
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<td>0.41</td>
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<td>0</td>
<td>2</td>
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<td>2.293</td>
<td>2.3993</td>
<td>94.66%</td>
</tr>
<tr>
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<td>2</td>
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<td>0.9</td>
<td>3.951</td>
<td>4.1061</td>
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</tr>
</tbody>
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**Conflict of interest**

Author declares that there is no conflict of interest.

**References**


